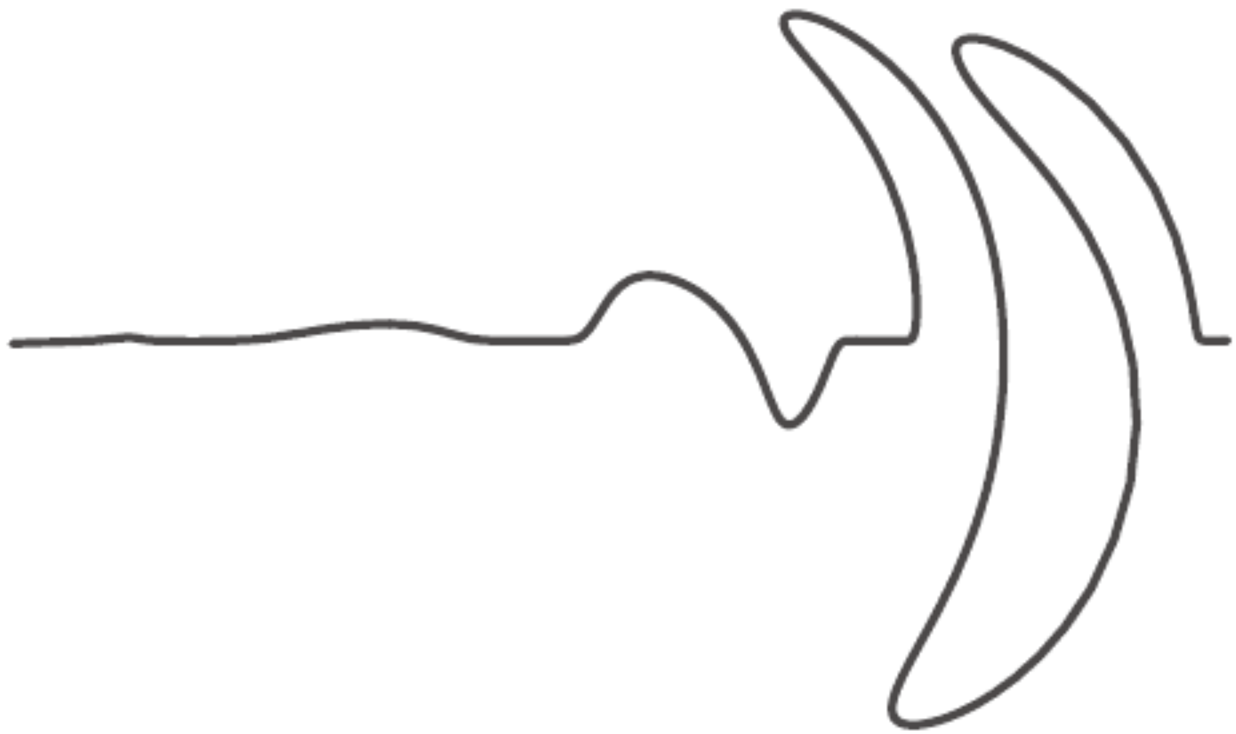
Bicycle "Unicycle" Path

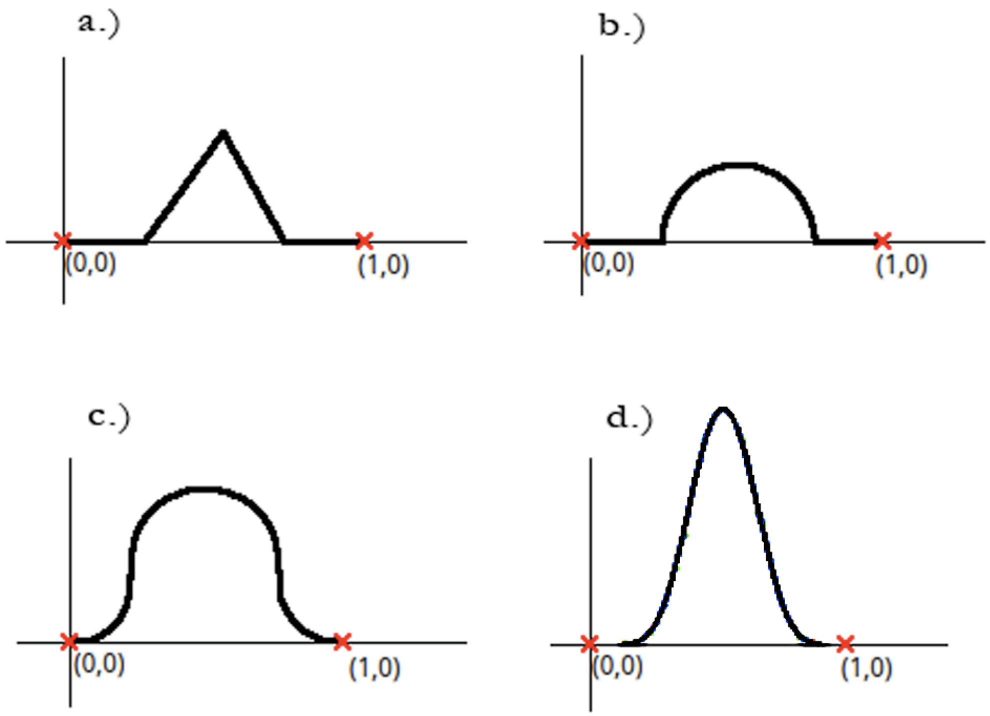


Finn, David. "Can a Bicycle Create a Unicycle Path"

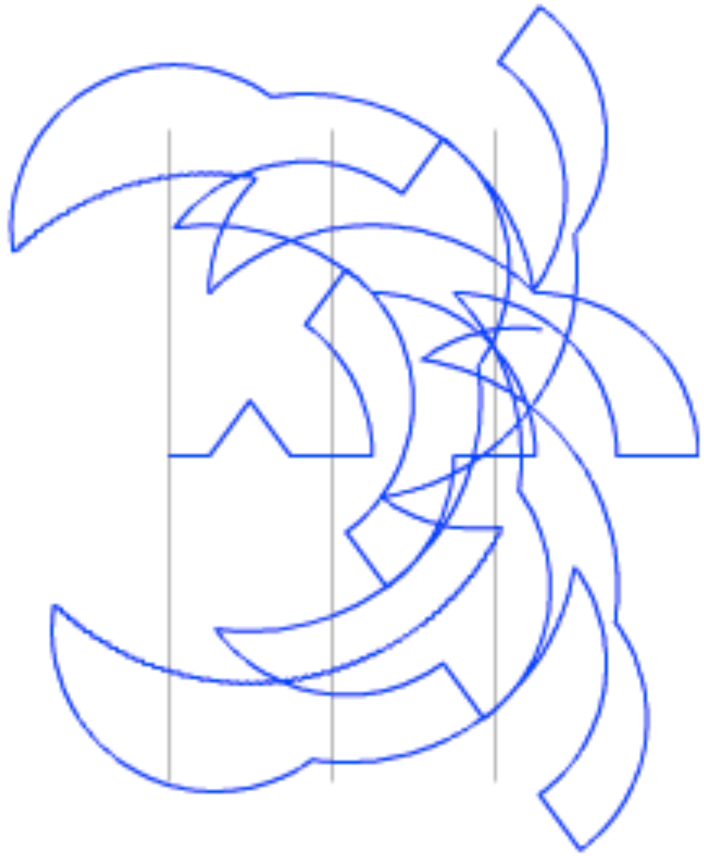
Definition of a *Seed Curve*

- The *seed curve* of a unicycle path is the curve from which the rest of the path is constructed. It is a piecewise smooth, vector-valued curve that with endpoints (0,0) and (1,0) with infinite zero derivative at each of these end points. It may self intersect and is not necessarily a graph.

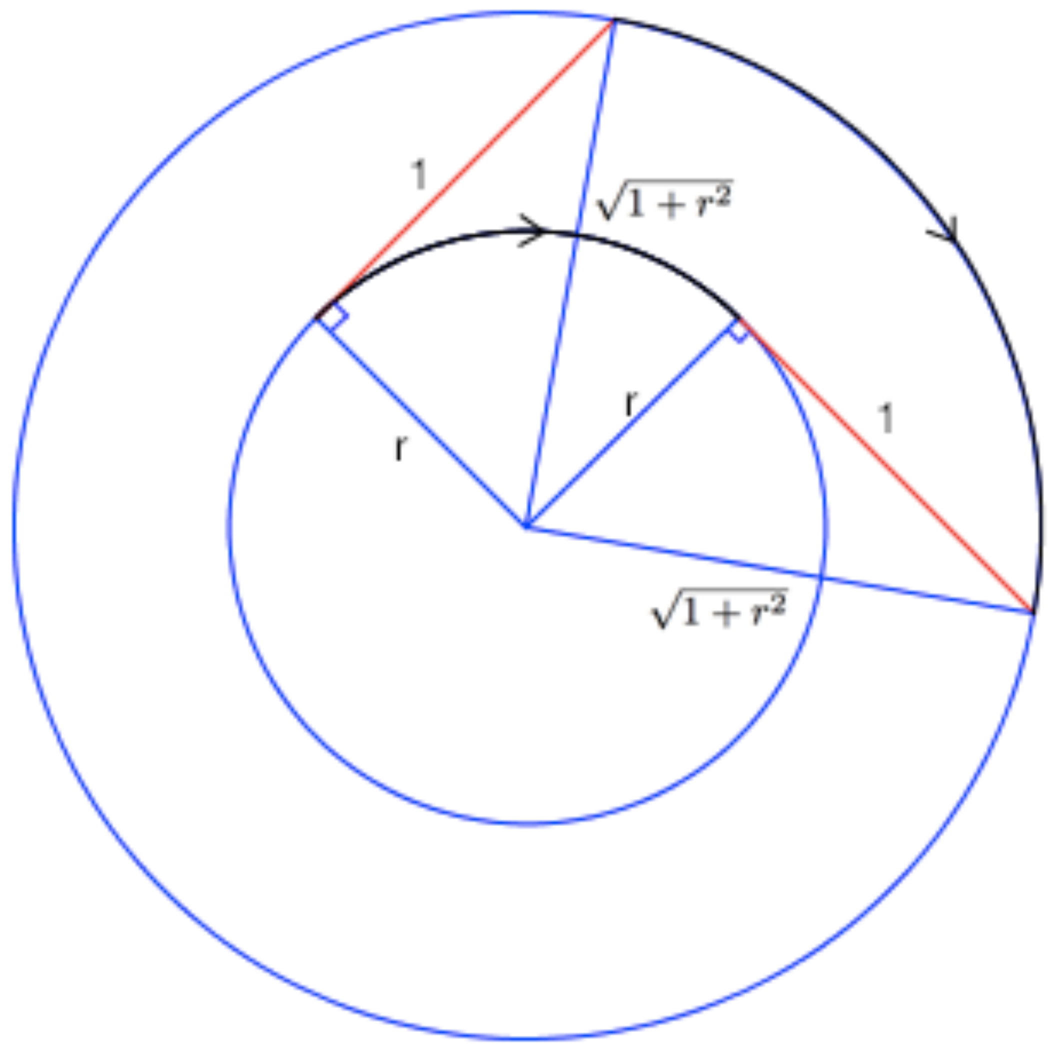
Various Discrete Seed Curves (a-c) and a Smooth Seed Curve (d)



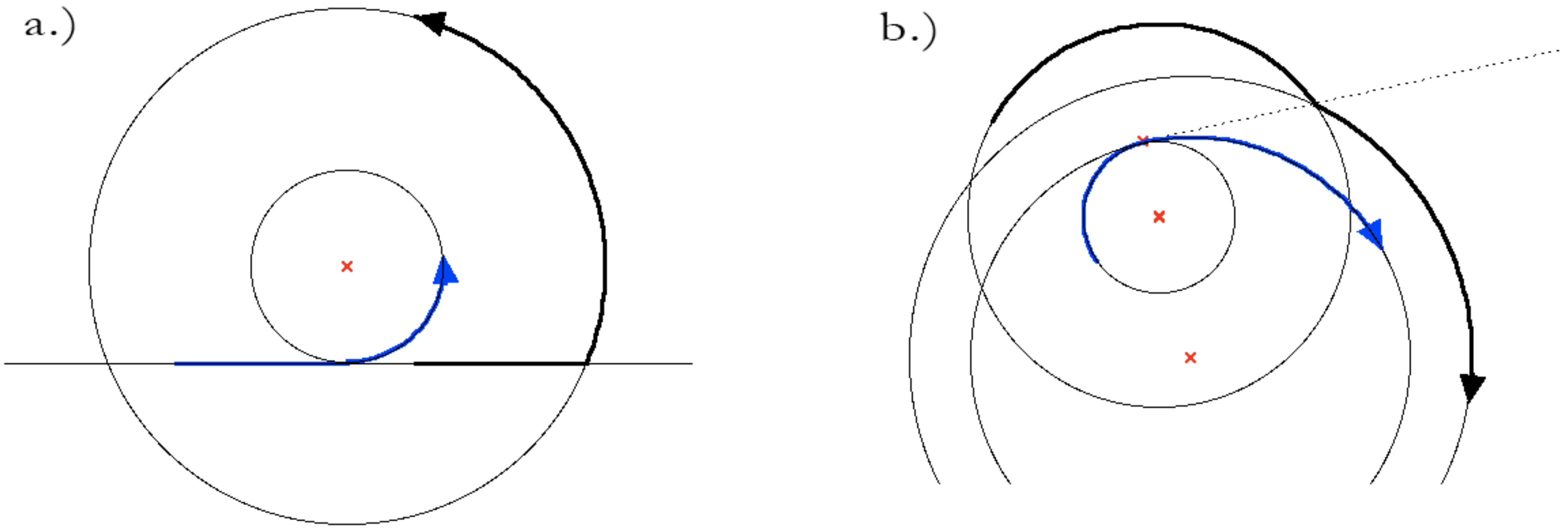
Propagation of a Straight Line Path

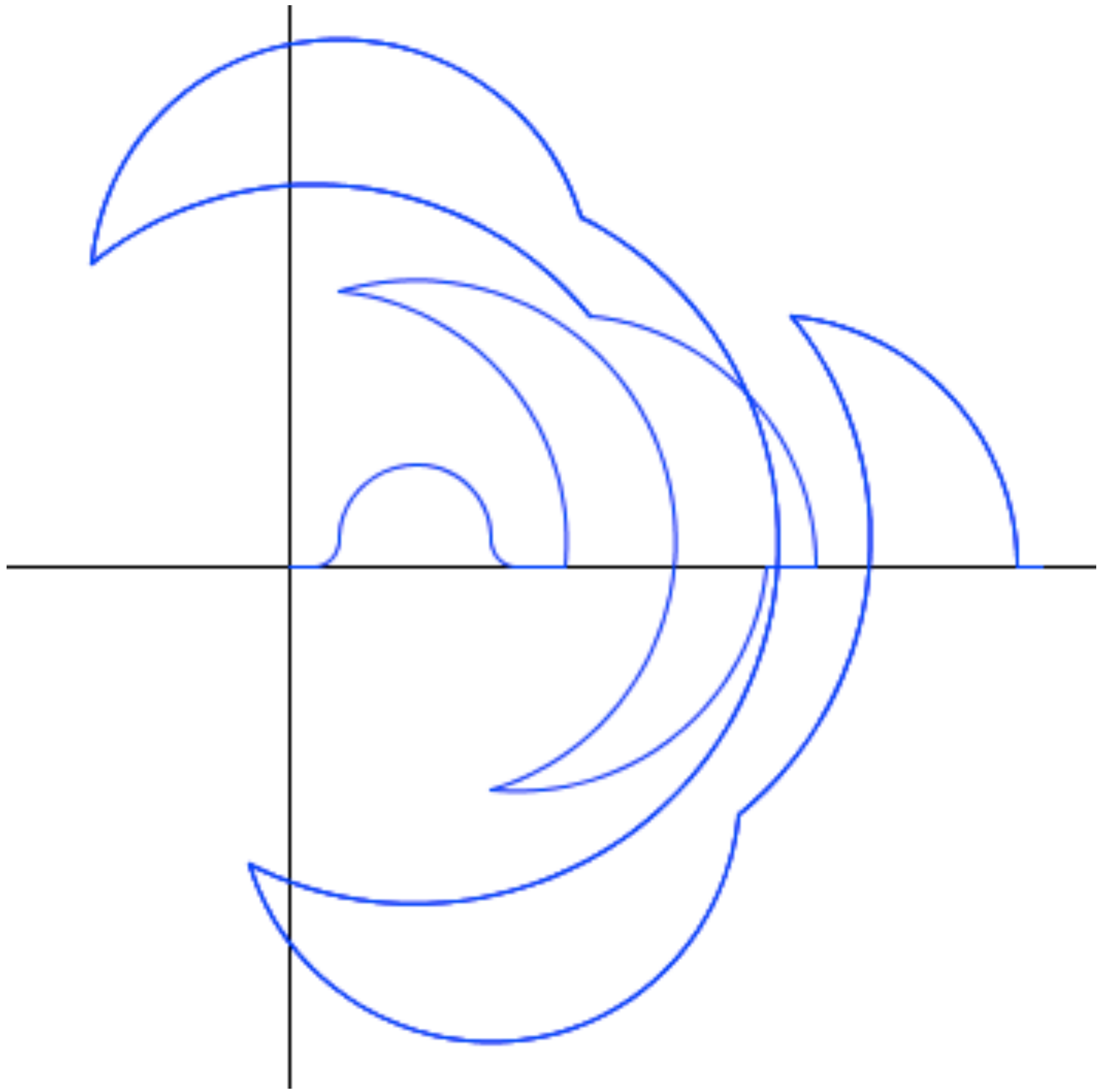


Concentric Circle Arc Bicycle Path

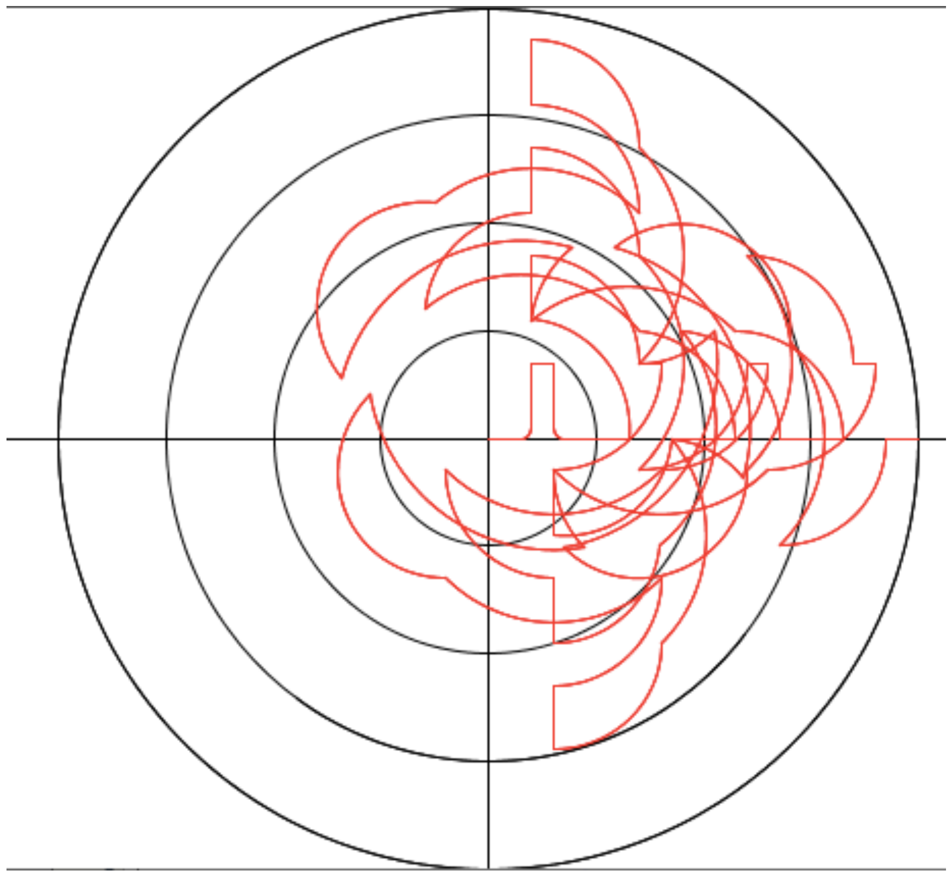


Result of a Continuous First Derivative

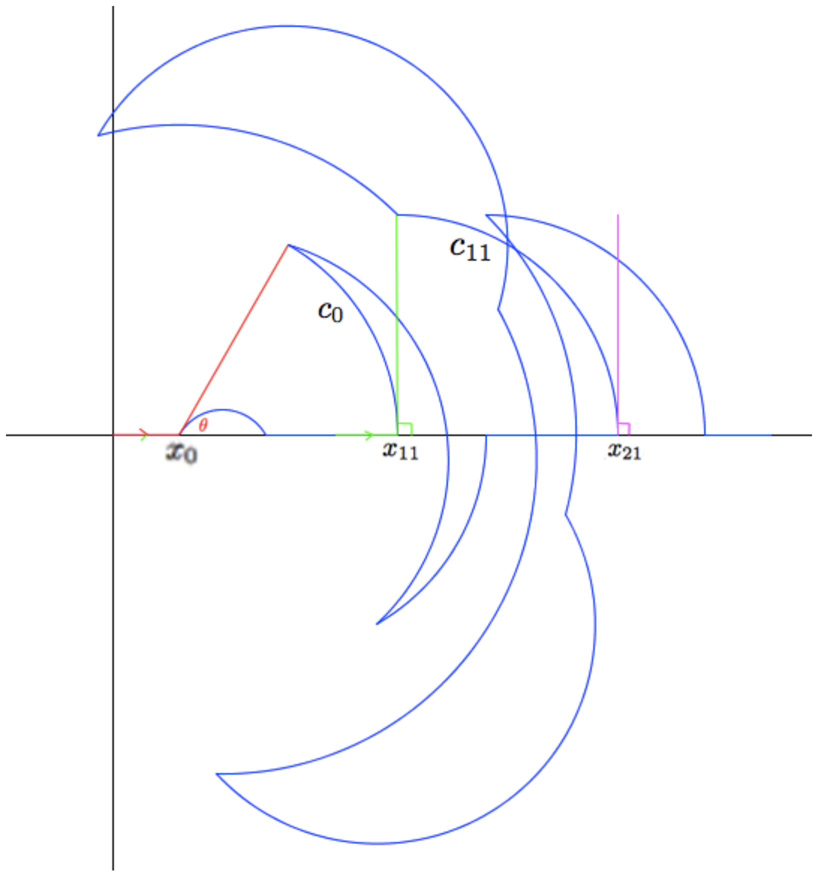




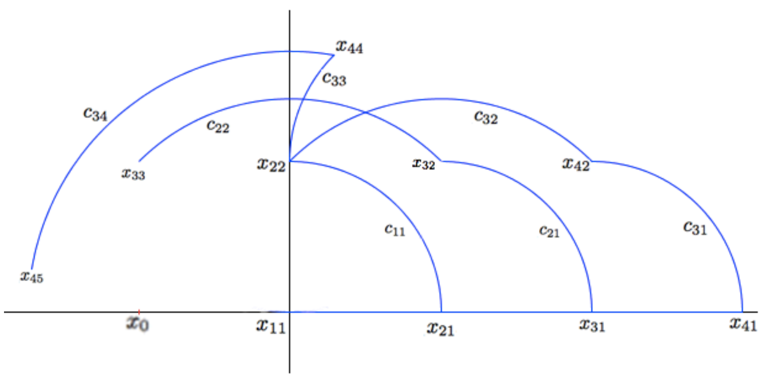
Linear Growth about the Origin in Radial Direction

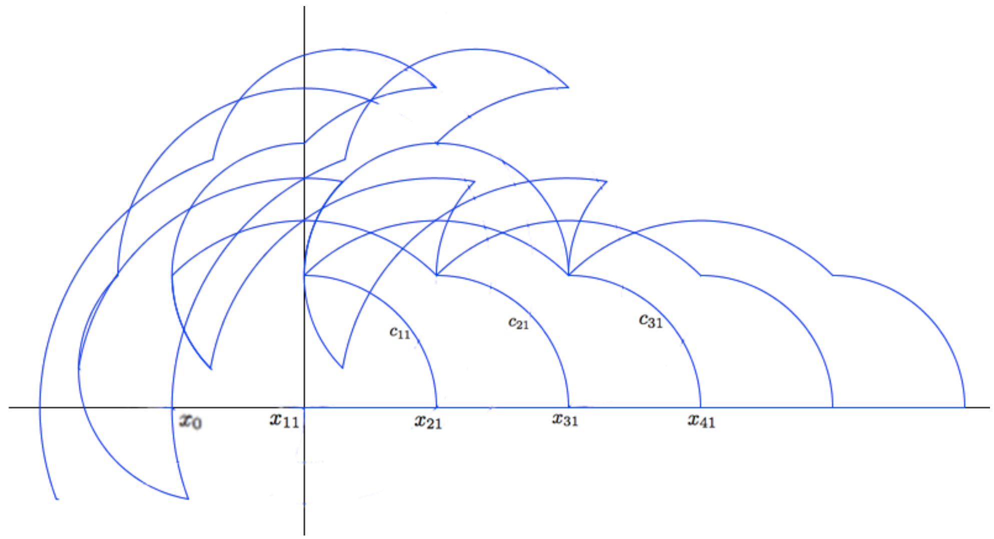


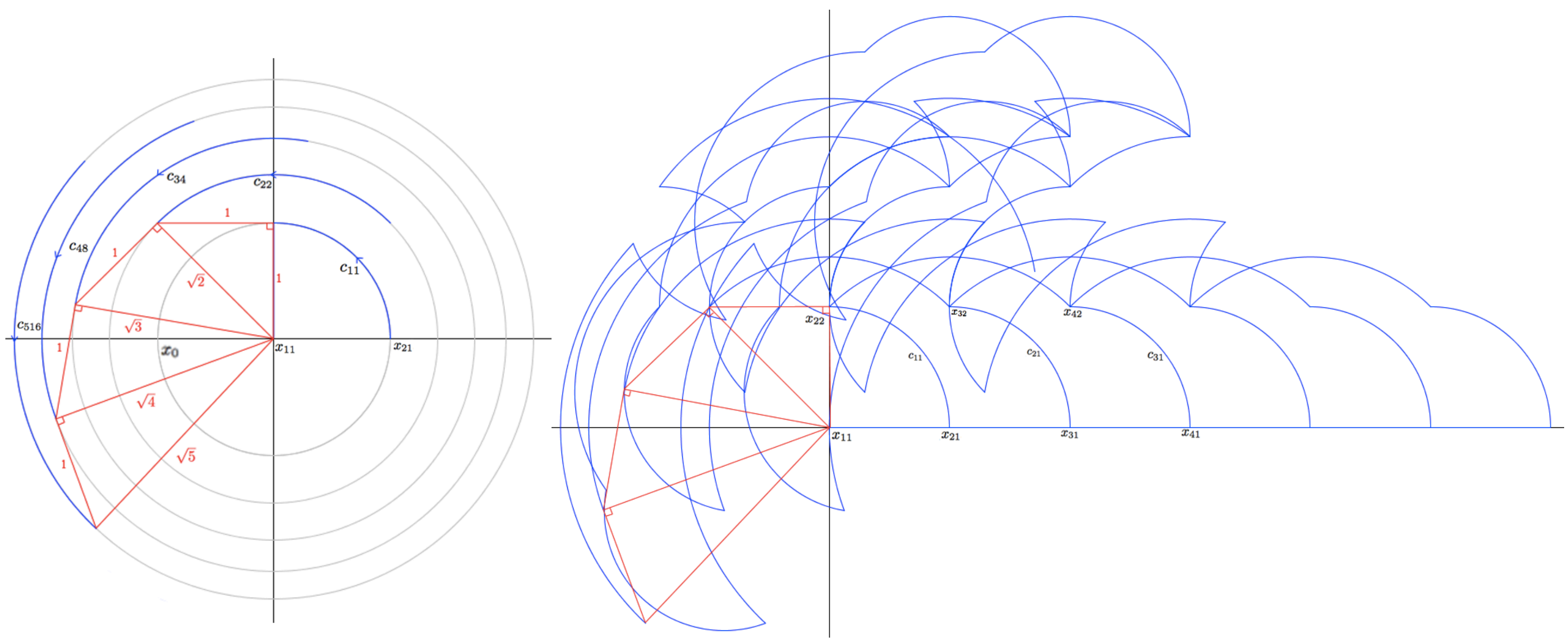
Growth About Arbitrary Discrete Seed Curve



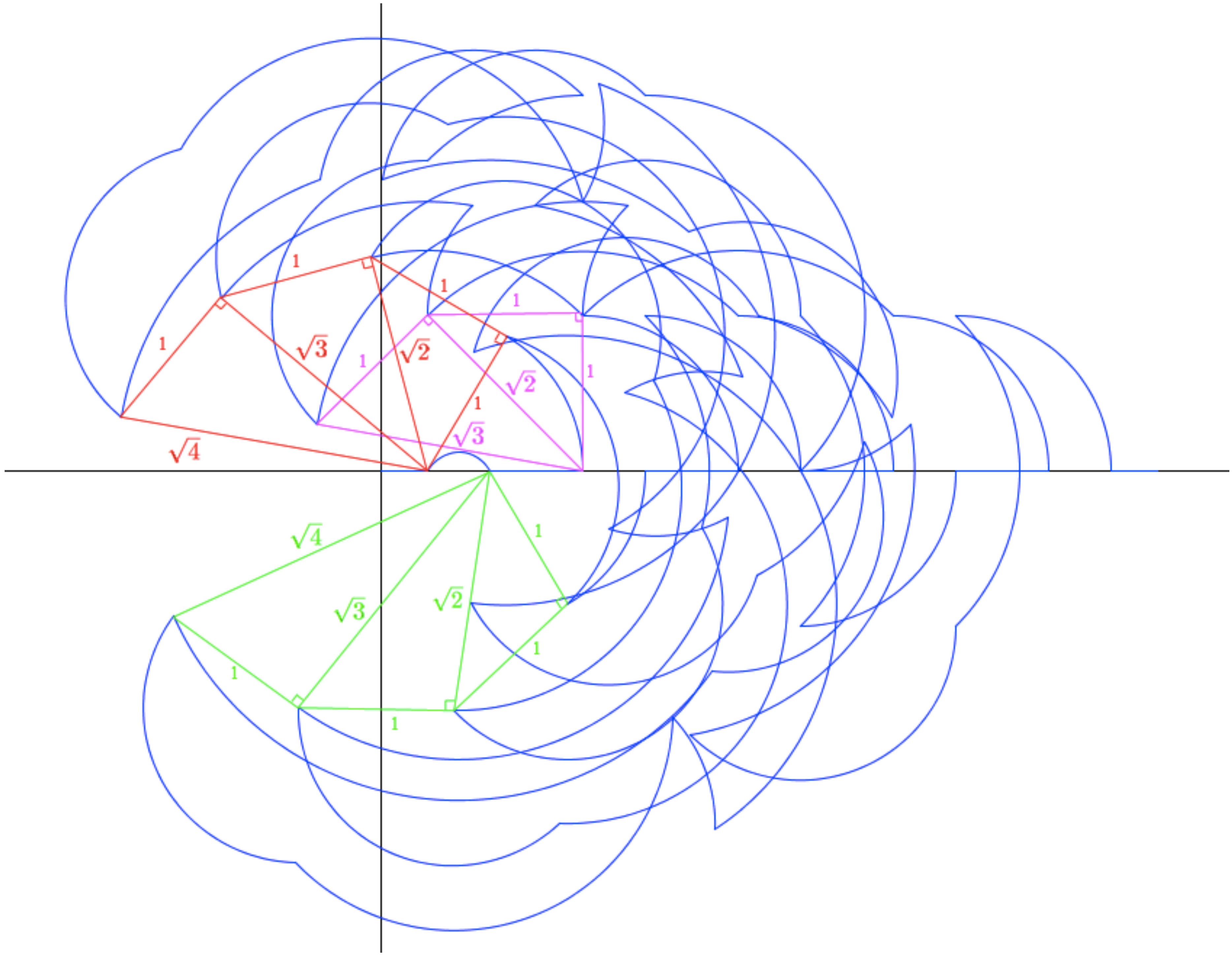
Invariant Portion of Discrete Bicycle Paths



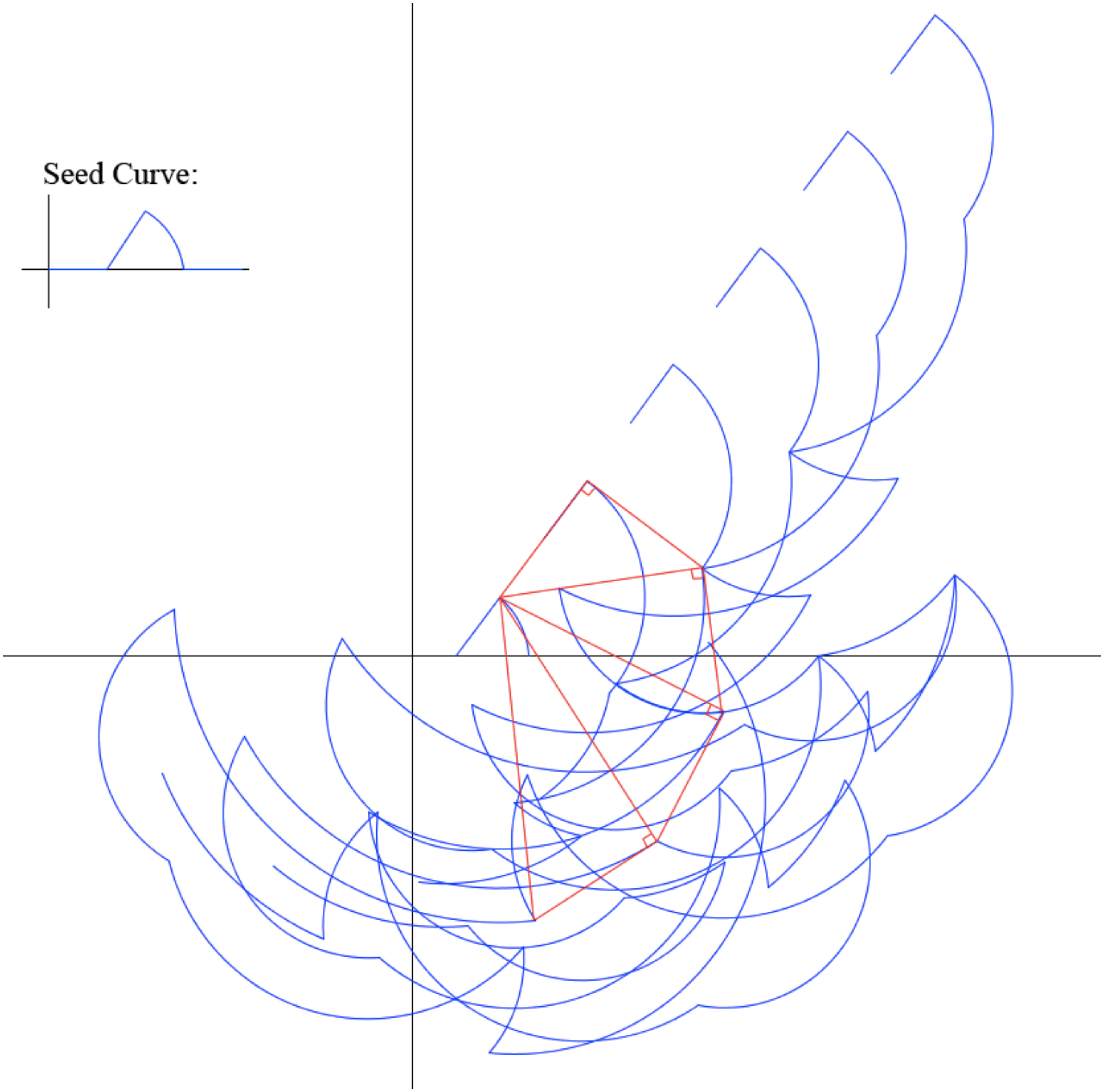


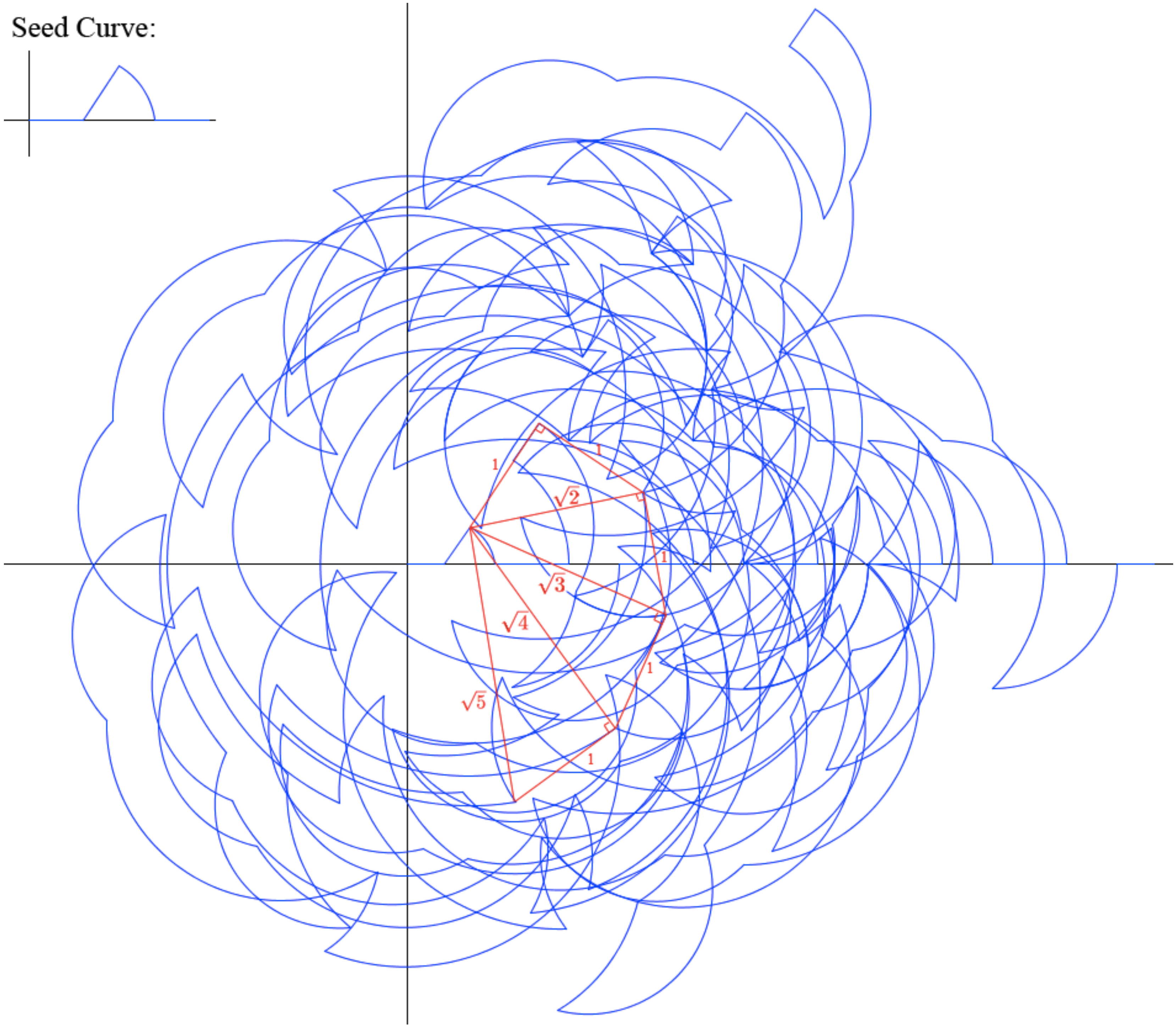


c_{34}
c_{22}
c_{48}
c_{11}
c_{516}
1
$\sqrt{2}$
$\sqrt{3}$
$\sqrt{4}$
$\sqrt{5}$
x_0
x_{11}
x_{21}
x_{22}
x_{32}
x_{42}
c_{11}
c_{21}
c_{31}
x_{11}
x_{21}
x_{31}
x_{41}



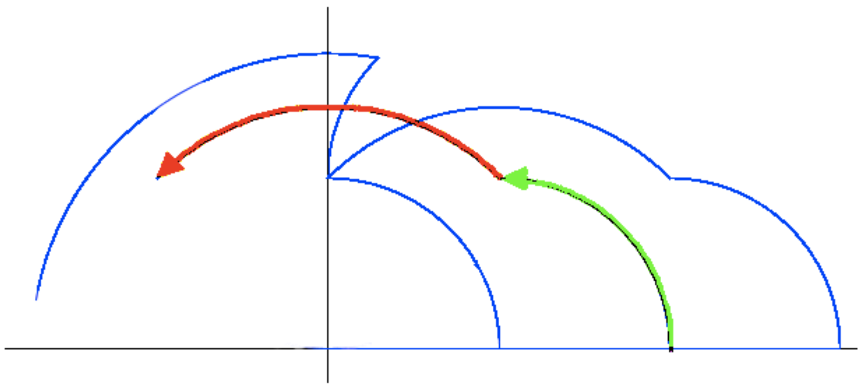
1
1
$\sqrt{3}$
$\sqrt{2}$
1
1
$\sqrt{2}$
1
1
$\sqrt{2}$
1
$\sqrt{3}$
$\sqrt{4}$
1
$\sqrt{4}$
$\sqrt{3}$
$\sqrt{2}$
1
1
1

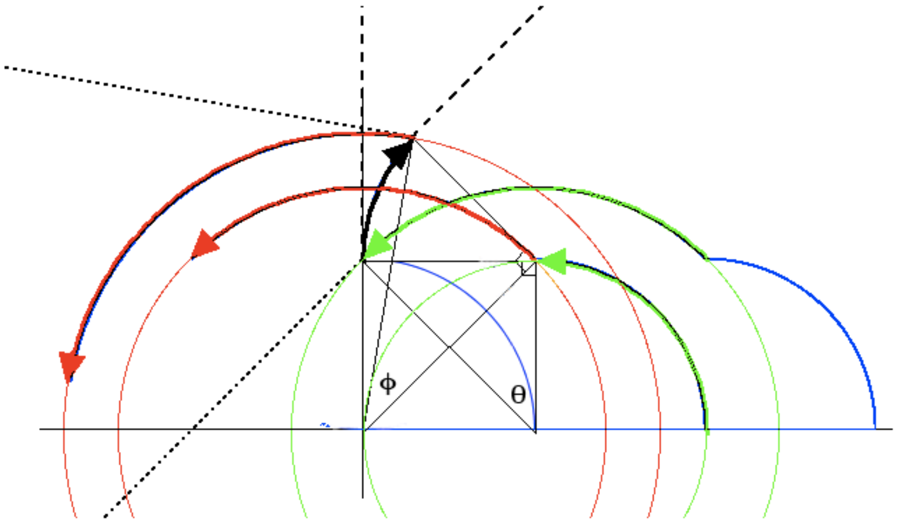

Seed Curve:



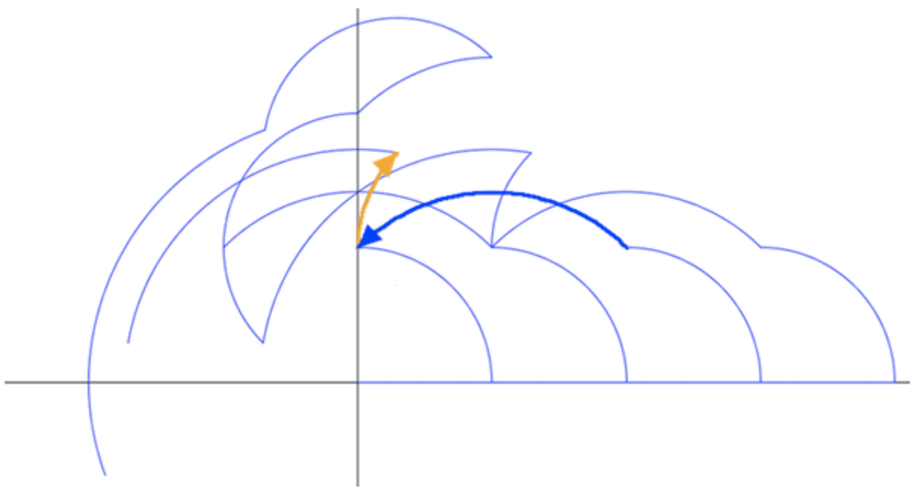
Seed Curve:
1
1
$\sqrt{2}$
1
$\sqrt{3}$
$\sqrt{4}$
1
$\sqrt{5}$
1

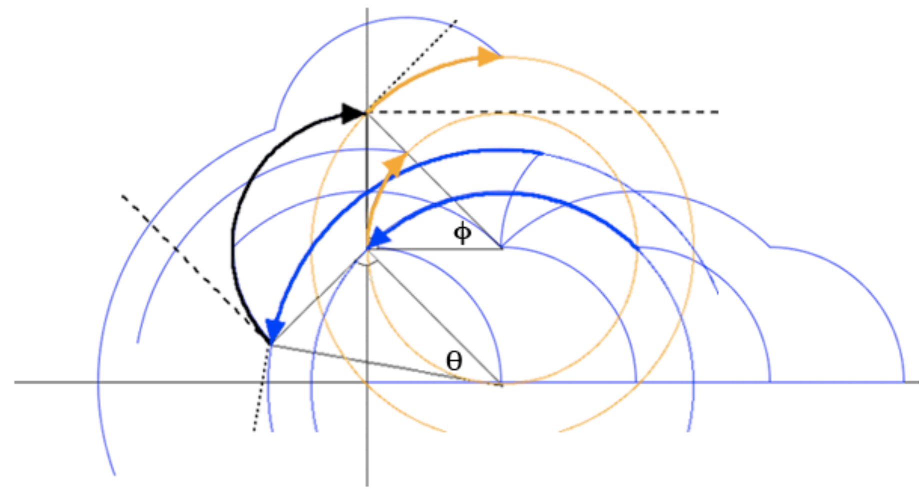
Result of Acute Corner





Result of Obtuse Corner





- An O in iteration *i* will result in OA in iteration *i+1*
- An A in iteration *i* will result in OO in iteration *i+1*
- Let A_n be the number of As in iteration *n* and let O_n be the number of Os in iteration *n*

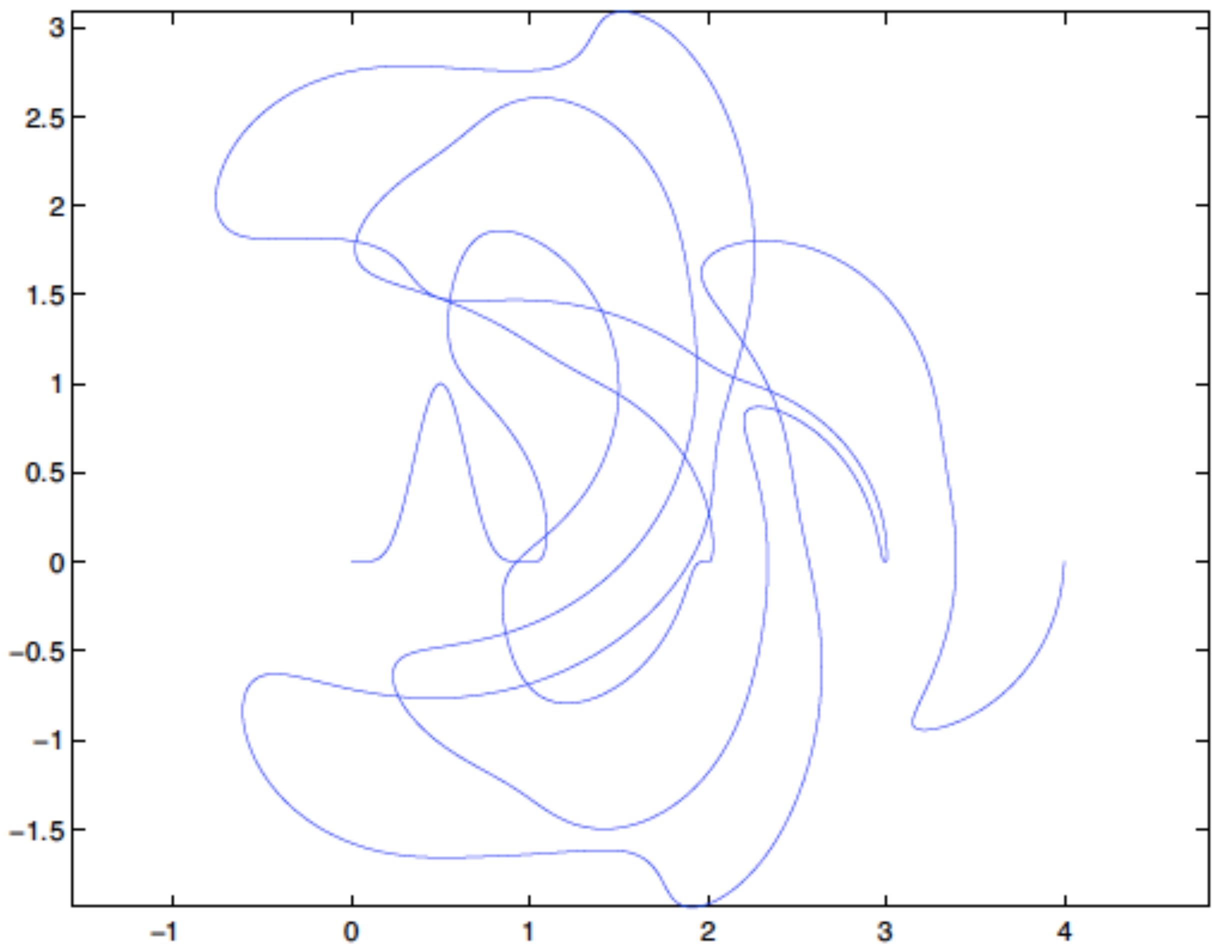
$$A \to OO \to OAOA \to OAOOOAOO$$

$$\begin{aligned} A_{n+1} &= O_n \\ O_{n+1} &= 2A_n + O_n \\ &= 2O_{n-1} + O_n \\ \implies 0 &= O_{n+1} - O_n - 2O_{n-1} \end{aligned}$$

$$\begin{aligned} 0 &= q^{n+1} - q^n - 2q^{n-1} \\ &= q^{n-1}(q^2 - q - 2) \\ &= q^2 - q - 2 \end{aligned}$$

$$\frac{4}{3}2^n + \frac{2}{3}(-1)^n$$

Smooth Model



Levi, Mark and Sergei Tabachnikov. On Bicycle Tire Track Geometry, Hatchet Planimeter, Menzin's Conjecture and Oscillation of Unicycle Tracks, arXiv:0801.4396,VOL. 1, 28 January 2008

Proposition 6 (**Levi, Tabachnikov**) *Denote by* $Z(\gamma_i)$ *the number of intersection points of the curve* γ_i *with the* x*-axis (excluding the end points of the iteration); assume that* $Z(\gamma_i)$ *is finite. One has* $Z(\gamma i + 1) > Z(\gamma i)$ *for any non-trivial smooth bicycle path [2].*

Proposition 7 (**Levi, Tabachnikov**) *Denote by* $E(\gamma_i)$ *the number of local extrema of the curve* γ_i*; assume that* $E(\gamma_i)$ *is finite. One has* $E(\gamma i + 1) > E(\gamma i)$ *for any non-trivial smooth bicycle path [2].*

We prove a similar proposition:

Proposition 8 *Denote by* $P(\gamma_i)$ *the number of points of inflection in the curve* γ_i*; assume that* $P(\gamma_i)$ *is finite. One has* $P(\gamma_{i+1}) > P(\gamma_i)$ *for any non-trivial smooth bicycle path.*

Solve for the Curvature of the i+1st iteration in terms of the curvature of the ith iteration:

$$\boldsymbol{\gamma_{i+1}} = \boldsymbol{\gamma_i} + \boldsymbol{\gamma_i'} \qquad \kappa_{i+1} = \frac{\boldsymbol{\gamma'_{i+1}} \times \boldsymbol{\gamma''_{i+1}}}{|\boldsymbol{\gamma'_{i+1}}|^3} = \frac{(\boldsymbol{\gamma'_i} + \boldsymbol{\gamma''_i}) \times (\boldsymbol{\gamma''_i} + \boldsymbol{\gamma'''_i})}{|\boldsymbol{\gamma'_i} + \boldsymbol{\gamma''_i}|^3}$$

Solving for each term:

$$\boldsymbol{\gamma''_i} = \kappa_i \mathbf{N} = \kappa \mathbf{B} \boldsymbol{\gamma'_i}$$

$$\begin{aligned} \boldsymbol{\gamma'''_i} &= (\kappa_i \mathbf{B} \boldsymbol{\gamma'_i})' \\ &= \kappa'_i \mathbf{B} \boldsymbol{\gamma'_i} + \kappa_i \mathbf{B} \boldsymbol{\gamma''_i} \\ &= \kappa'_i \mathbf{N} + \kappa_i^2 \mathbf{BN} \\ &= \kappa'_i \mathbf{N} - \kappa_i^2 \boldsymbol{\gamma'_i}. \end{aligned}$$

$$\begin{aligned} |\boldsymbol{\gamma'_{i+1}}| &= \sqrt{(\boldsymbol{\gamma'_i} + \boldsymbol{\gamma''_i}) \cdot (\boldsymbol{\gamma'_i} + \boldsymbol{\gamma''_i})} \\ &= \sqrt{1 + \kappa_i^2}. \end{aligned}$$

Some Math…

$$\begin{aligned}
\kappa_{i+1} &= \frac{(\boldsymbol{\gamma}_i' + \boldsymbol{\gamma}_i'') \times (\boldsymbol{\gamma}_i'' + \boldsymbol{\gamma}_i''')}{|\boldsymbol{\gamma}_i' + \boldsymbol{\gamma}_i''|^3} \\
&= \frac{(\boldsymbol{\gamma}_i' + \kappa_i \mathbf{N}) \times (\kappa_i \mathbf{N} + \kappa_i' \mathbf{N} - \kappa_i^2 \boldsymbol{\gamma}_i')}{(\sqrt{1 + \kappa_i^2})^{3/2}} \\
&= \frac{\kappa_i + \kappa_i' + \kappa_i^3}{(1 + \kappa_i^2)^{3/2}}.
\end{aligned}$$

Let F be an antiderivative of $1 + \kappa_i^2$. Than one has,

$$\begin{aligned}
\kappa_i + \kappa_i' + \kappa_i^3 &= e^{-F}(e^F \kappa_i)' \\
&= F' \kappa_i + \kappa_i'.
\end{aligned}$$

Definition A *seed oval*, α_0, is convex, vector-valued, closed curve from which the rest of the path is constructed.

Conjecture 1 If α_0 is not a circle then $\exists i \in \mathbb{N}$ such that α_i is not convex.