Gutkin's Problem in Constant Curvature Geometries and Discrete Version

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August 8th, 2013

Gutkin's Theorem in E^2

Given a smooth, convex and closed curve γ, assume that two points, X and Y, can "chase" each other around γ in such a way that the angle made by the chord $\overline{XY}$ with γ at both end points has a constant value, say, α. If γ is not a circle, what are possible values of α?

The answer is as follows: a necessary and sucient condition is that

Theorem

There exists integer $k \geq 2$ such that $k \tan \alpha = \tan(k\alpha)$.

Motivation and Goal

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- Billiards problem
- Rigidity theory
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Goal

- Develop similar behavior in discrete case
- Develop similar behavior in other constant curvature geometries

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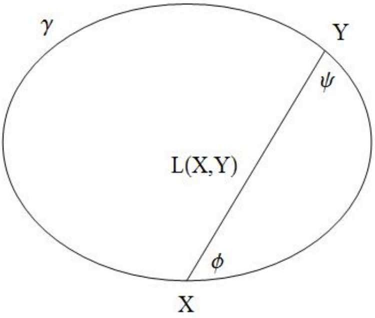

Figure : Curve γ with chord xy

What we know:

- γ is a convex, smooth, and closed curve.
- $X = \gamma(x(t))$ and $Y = \gamma(y(t))$ are two points on γ.
- The angles made by chord $\overline{XY}$ and γ are constant for all t.
- $x(t), y(t) \in [0, L(\gamma)]$ are arc length parameters of γ.

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Define:

- $L(x, y)$ be the arc length of chord $\overline{XY}$.

Proposition

We should choose t in a way such that $x_t = a/\kappa(x)$ and $y_t = a/\kappa(y)$, where a is a constant.

Proof.

We have following equations from Bialy's paper:

$$L_x = -\cos\phi, L_y = \cos\psi$$

$$L_{xx} = \frac{\sin^2\phi}{L} - \kappa(x)\sin\phi \quad L_{yy} = \frac{\sin^2\psi}{L} - \kappa(y)\sin\psi$$

$$L_{xy} = \frac{\sin\phi\sin\psi}{L}$$

We want ϕ and ψ to be a constant α. Replacing α into above equations and since α is a constant, we have that:

$$0 = L_{xt} = L_{xx}x_t + L_{xy}y_t$$

$$0 = L_{yt} = L_{xy}x_t + L_{yy}y_t$$

Then we can compute that

$$\frac{y_t}{x_t} = \frac{\kappa(x)}{\kappa(y)}.$$

Remark

We know that $0 \leq t \leq T$, where T is the upper bound of t. We can choose a to make T to be 2π for later computation.

Proposition

Set $f(t+c)$ *and* $f(t-c)$ *be such that* $f(t+c) = \frac{\sin\alpha}{\kappa(x)}$ *and* $f(t-c) = \frac{\sin\alpha}{\kappa(y)}$. *Then we have that*

$$f'(t+c) + f'(t-c) = a\cot\alpha\left(f(t+c) - f(t-c)\right).$$

Proof.

From former equations, we can solve for L that

$$L = \frac{\sin\alpha}{\kappa(x)} + \frac{\sin\alpha}{\kappa(y)}.$$

Then we have that

$$L = f(t+c) + f(t-c).$$

It follows that $\dot{L} = \dot{f}(t+c) + \dot{f}(t-c)$, where $\dot{f}$ denotes derivative of f respect to t. By chain rule, we have that

$$\dot{L} = L_x\dot{x} + L_y\dot{y} = \frac{a\cos\alpha}{\sin\alpha}(f(t+c) - f(t-c)).$$

Therefore,

$$f'(t+c) + f'(t-c) = a\cot\alpha\,(f(t+c) - f(t-c))\,.$$

□

Since $f(t)$ is a function with period of 2π, thus using Fourier expansion, we have that $f(t) = \quad b_k e^{ikt}$, where $b_k \in \mathbb{C}$, and $b_{-k} = \overline{b_k}$. It follows that

$$f(t \pm c) = \quad b_k e^{\pm ikc} e^{ikt}$$

$$f'(t \pm c) = \quad b_k ike^{\pm ikc} e^{ikt}.$$

Plugging into former proposition and equating both sides, we get:

$$k \tan \alpha = a \tan kc$$

Remark

- *$a = 1$ in E^2 by Gauss-Bonnet Theorem.*
- *$k \geq 2$ in E^2 by Fourier analysis.*
- *$c = \alpha$ in E^2*

Spherical version of Gutkin's theorem

In S^2, the computation is much more complicated for a general curve. We know that circle is always a solution to this problem. Therefore, we start from a circle γ_0 with radius R. Then, we are going to deform γ_0 and find infinitesimal solutions in a small neighborhood of the circle.

Spherical version of Gutkin's theorem

Theorem

Given a smooth, convex and closed curve γ which can be obtained by deforming a circle, assume that two points, X and Y, can "chase" each other around γ in such a way that the angle made by the chord $\overline{XY}$ with γ at both end points has a constant value, say, α. One sufficient and necessary condition is that

$$\tan(kc) = k\tan(c)\frac{1}{a\sqrt{\cos^2 c\tan^2 R + 1}},$$

where k is an integer with $k \geq 2$, R is the radius of the circle we deform and $\cos\alpha = \frac{\cos c}{\sin^2 R \sin^2 c + \cos^2 R}$.

New Equations

Bialy's equations in S^2:

$$L_x = -\cos\alpha, L_y = \cos\alpha$$

$$L_{xx} = \frac{\sin^2\alpha}{\tan L} - \kappa(x)\sin\alpha \quad L_{yy} = \frac{\sin^2\alpha}{\tan L} - \kappa(y)\sin\alpha$$

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Bialy's equation in E^2:

$$L_x = -\cos\phi, L_y = \cos\psi$$

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We should choose parameter t in a way such that $x_t = a/\sqrt{\kappa(x)^2 + \sin^2\alpha}$, *where a is a constant to make* $T = 2\pi$.

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Proposition

We should choose t in a way such that $x_t = a/\kappa(x)$ and $y_t = a/\kappa(y)$, where a is a constant.

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Let $f(t+c)$ *and* $f(t-c)$ *be such that* $\tan f(t+c) = \frac{\sin\alpha}{\kappa(x)}$ *and* $\tan f(t-c) = \frac{\sin\alpha}{\kappa(y)}$. *Then*

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New Propositions

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- $\sin f$ is involved in the equation and therefore cannot solve it as in E^2.
- Circle is always a solution to this question.
- Start from a circle γ_0 with radius R and deform it to find infinitesimal solutions.
- $f_0 = \cot^{-1}\left(\frac{\cot R}{\sin \alpha}\right)$, which is a constant.

Then we deform the circle infinitesimally with $f(t) = f_0 + \epsilon g(t)$ and $c = c + \epsilon\delta$, keeping α fixed.It follows that

$$g'_+ + g'_- = a \cot\alpha \left(\sin(f_0 + \epsilon g_+) - \sin(f_0 + \epsilon g_-) \right).$$

Applying Taylor expansion to $\sin(x + \epsilon y)$ and ignoring terms which has ϵ power higher than 2, we have that $\sin(x + \epsilon y) = \sin x + \epsilon y \cos x$. Then

$$g'_+ + g'_- = a \cot\alpha \cos f_0 \left(g_+ - g_- \right).$$

Solving the equation in S^2

Doing similar Fourier expansion as in the Euclidean case, we find that

$$\tan kc = \frac{k \tan \alpha}{a \cos f_0} = \frac{k \tan \alpha}{a \cos[\cot^{-1}(\frac{\cot R}{\sin \alpha})]} = \frac{k}{a} \tan \alpha \sqrt{\tan^2 R \sin^2 \alpha + 1},$$

where k is an integer.

Lemma

In S^2, α and c satisfy the following equation:

$$\cos\alpha = \frac{\cos c}{\sqrt{\sin^2 R \sin^2 c + \cos^2 R}}$$

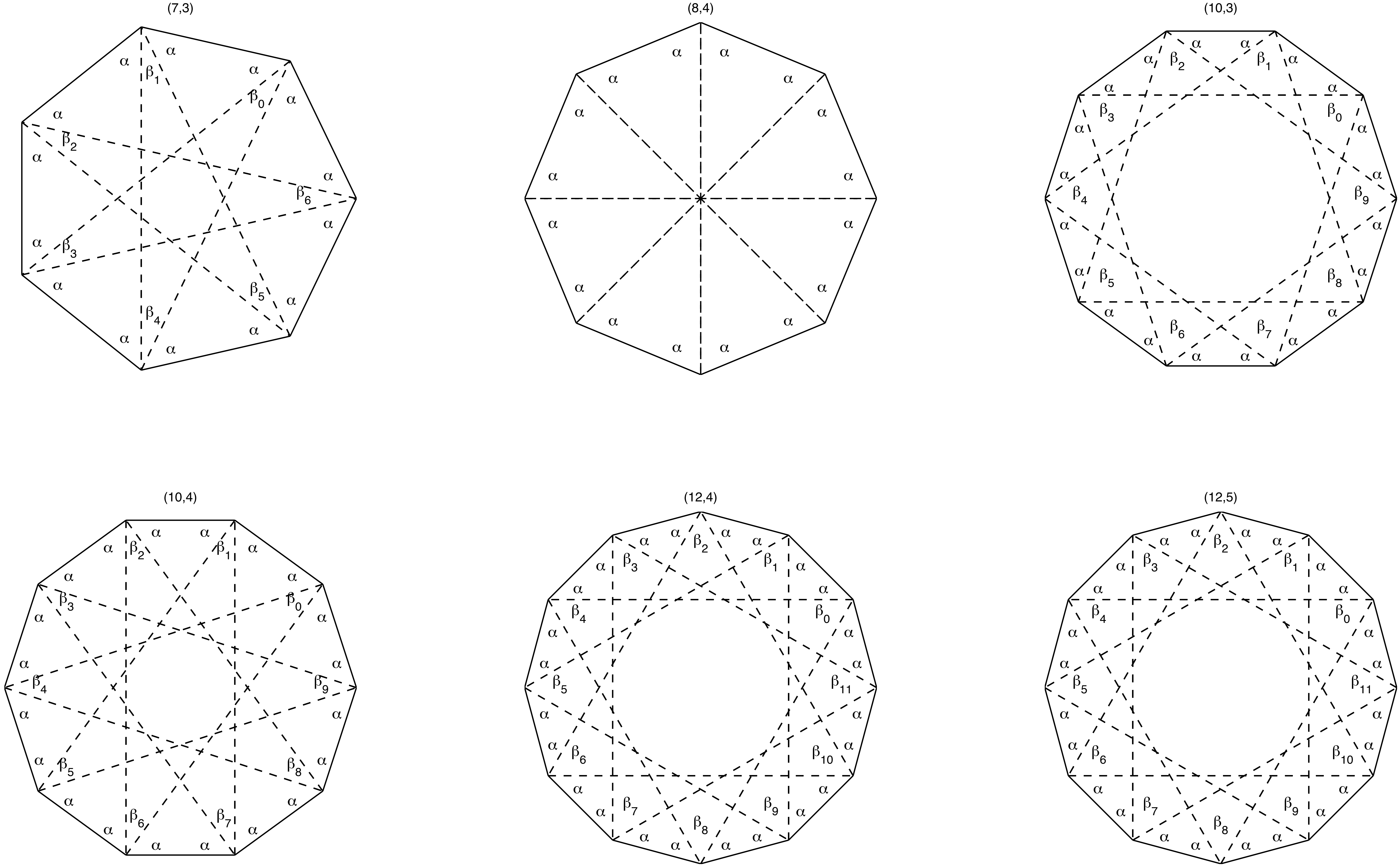
Analogous formula in S^2

Applying the lemma to former equation, we have that

$$\tan kc = k \tan c \frac{1}{a\sqrt{\cos^2 c \tan^2 R + 1}}.$$

Remark

Taking limit when R approaches 0, we have that $\tan(kc) = \frac{k}{a} \tan c$. We know that $T = \frac{1}{a} \int_0^{L(\gamma)} \sqrt{\kappa(x)^2 + \sin^2 \alpha} \, \mathrm{d}x = 2\pi$. When R approaches 0, $\kappa(x)$ approaches infinity, and therefore we have that $T = \frac{1}{a} \int_0^{L(\gamma)} \kappa(x) \, \mathrm{d}x = 2\pi$. On the unit sphere, when R approaches 0, it is relatively small to the sphere and locally conformal to E^2. Therefore, by Gauss-Bonnet theorem again we have that $a = 1$ and then we obtain the same formula $\tan kc = k \tan c$ as in E^2.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