Outer Billiard on Piecewise Circular Curves & Piecewise Hyperbola Curves

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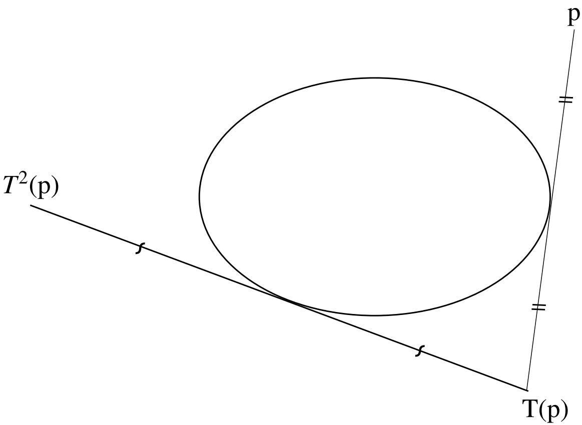
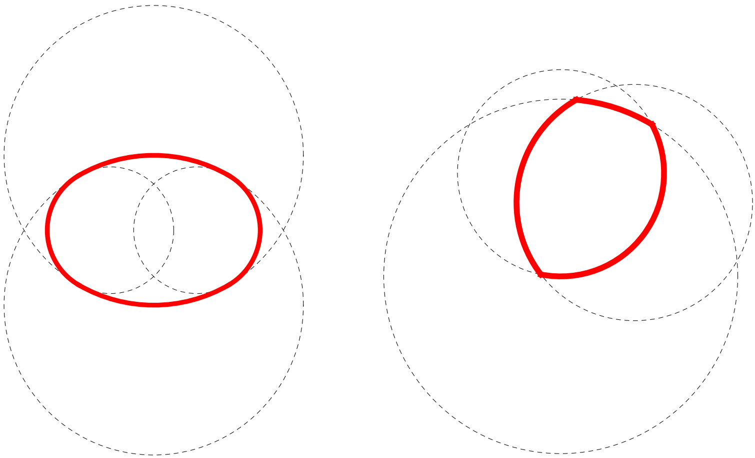
Outer Billiard

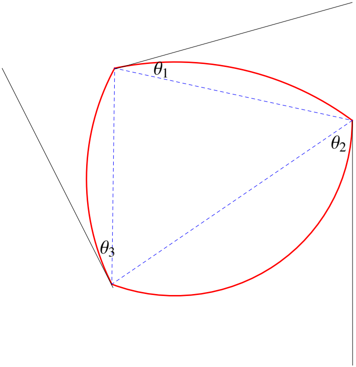


Piecewise Circular Curves



Definition

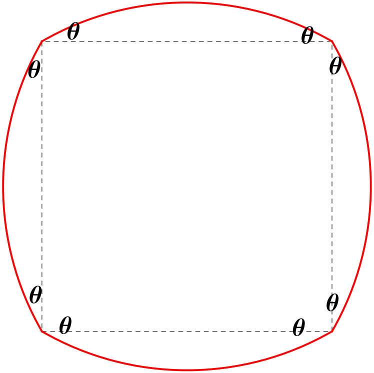
Piecewise circular curve is a curve defined by a polygon and angles with respect to each side of polygon.



Regular Piecewise Circular Curves

Definition

Regular piecewise circular curve is a piecewise circular curve such that all arcs are constructed from the same radius



Previous Studies - Sufficiently Smooth Table

Theorem (Douady)

If ∂K is at least C^6 smooth and positively curve, then all outer billiard orbits on K are bounded

Theorem (Birkhoff)

If ∂K is at least C^6 smooth and positively curve, outer billiard map about K has at least two n-periodic of rotation number r for any n, r relatively prime and $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$

Previous Studies - Polygonal Table

Theorem (Kolodziej)

All orbits in quasi-rational polygon are bounded

Theorem (Culter)

All convex n-gon has a periodic trajectory of rotation number $\lfloor \frac{n-1}{2} \rfloor$

Theorem (Schwartz)

Outer billiards on the Penrose kite has an unbounded orbit

Outline

1. Periodic Orbits
2. Preserved Region
3. Vector Fields at Infinity
4. Polygonal Invariant Curves
5. Rotation Numbers

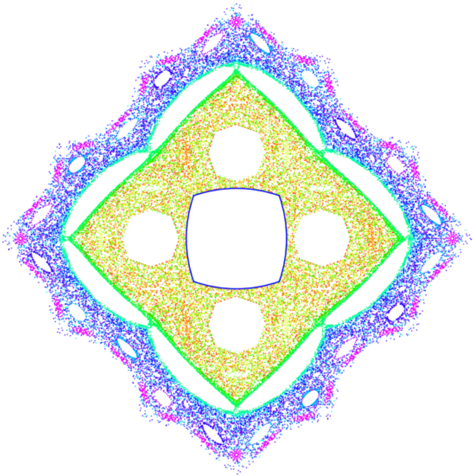


Figure : 50 orbits of outer billiard orbit about regular PC table constructed from a square of unit length and circles of radius 1.5

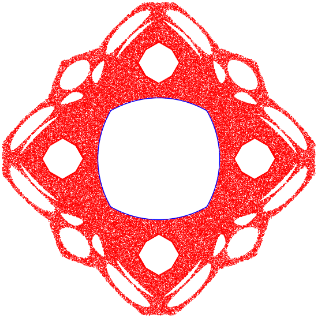


Figure : An orbit of outer billiard orbit about regular PC table constructed from a square of unit length and unit circles

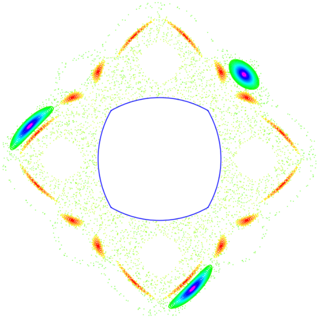
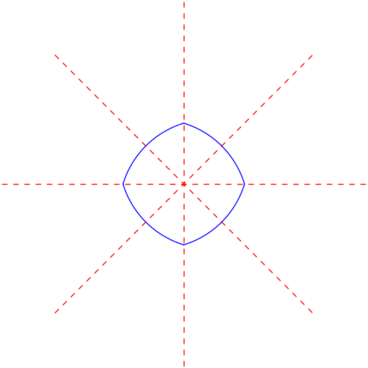


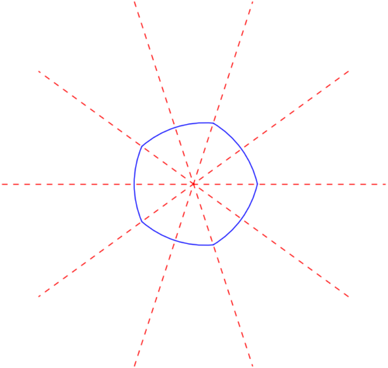
Figure : 30 orbits of outer billiard orbit about regular PC table constructed from a square of unit length and unit circles

Periodic Orbits

Theorem

Suppose P is a table with continuous outer billiard map T such that P has at least one reflection symmetry. For any $n \geq 3$, $\lfloor (n-1)/2 \rfloor \geq r \geq 1$ such that n and r are relatively prime, there exist a n-periodic orbit of rotation number r.





For regular PC curves constructed from m-gon, there exist at least $\frac{m}{(m,n)}$ n-periodic orbits of rotation number k if m, n are relatively prime.

Theorem

For any strictly convex table P, the circumscribed polygon that gives minimal area form a periodic trajectory.

Corollary

For any relatively prime pair n, r such that $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, outer billiard map about piecewise circular curves has at least one n-periodic orbit of rotation number.

Periodic Orbits

Theorem (Birkhoff)

For any sufficiently smooth curve (C^6) *P, there exist at least two n-periodic of rotation number r for any* n, r *relatively prime and* $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$

Open Question

Is there a similar result for Birkhoff's theorem for piecewise circular curves? More generally, what about strictly convex tables?

Conjecture: Yes

Periodic Orbits

Proposition

For any piecewise circular curves constructed from equilateral triangle (not necessary symmetry), there exist at least two 3-periodic orbits.

Proposition

Suppose P is a piecewise circular curve constructed from triangle ABC whose angles of tangent lines on sides BC, AC, AB are $\theta_A, \theta_B, \theta_C$ respectively. If $\theta_A \leq \min\{\hat{B}, \hat{C}\}$, $\theta_B \leq \min\{\hat{A}, \hat{C}\}$, $\theta_C \leq \min\{\hat{A}, \hat{B}\}$, outer billiard about P has at least two 3-periodic orbits.

Preserved Region

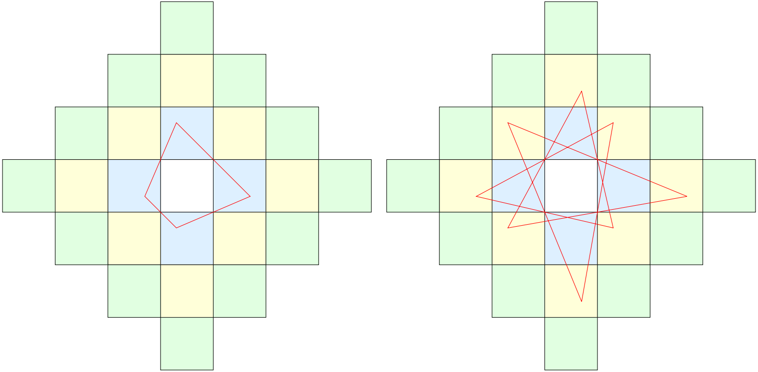
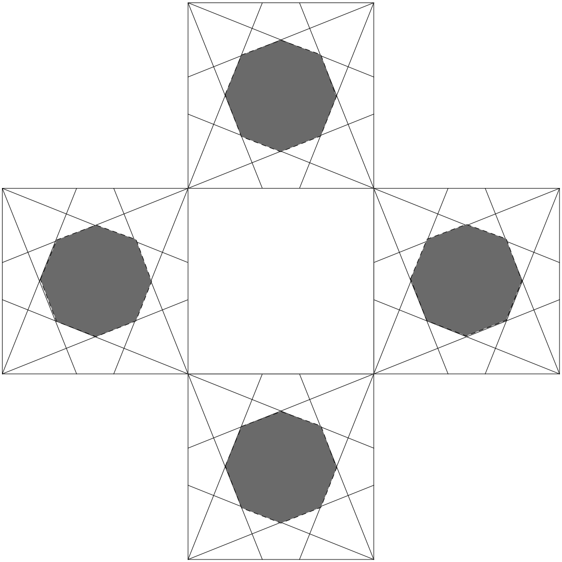


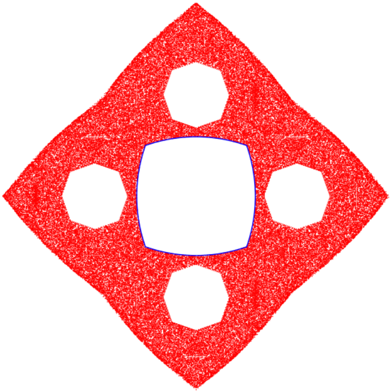
Figure : All points outside a square are periodic orbits under outer billiard map



Preserved Region

Definition

Preserved regions are closed region such that all points are periodic under outer billiard map



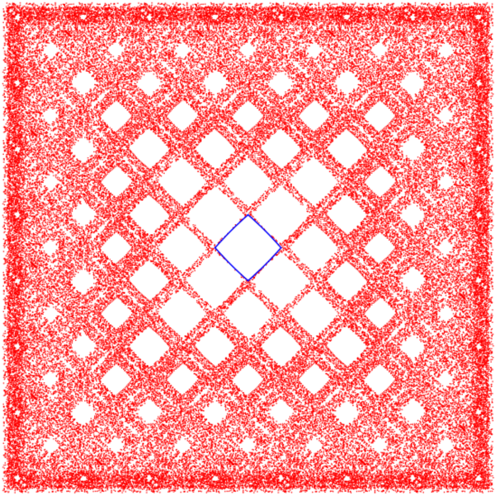


Figure : An orbit of outer billiard about regular PC curve constructed from a unit square and circles of radius $5\sqrt{2}(\theta = \arctan\left(\frac{1}{10\sqrt{2}}\right))$

Preserved Region

Proposition

For a regular PC table constructed from a square of unit side length with tangent angle θ, preserved region at k-th layer exist if and only if $\tan\theta \leq \frac{1}{2k-1}$. If so, the area of the region is given by formula

$$\frac{\left((2k-1)\sin\theta - \cos\theta\right)^2}{2\sin^2\left(\pi/4+\theta\right)}$$

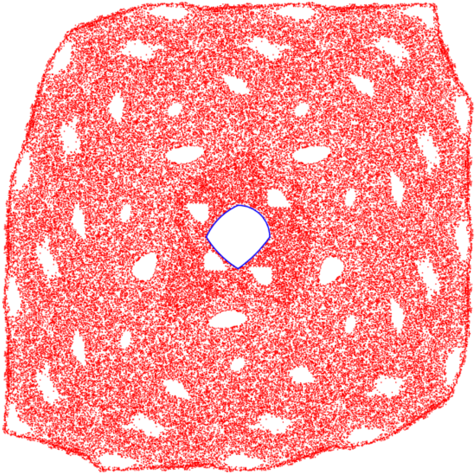


Figure : Non-symmetric preserved region of outer billiard about PC curve constructed from square

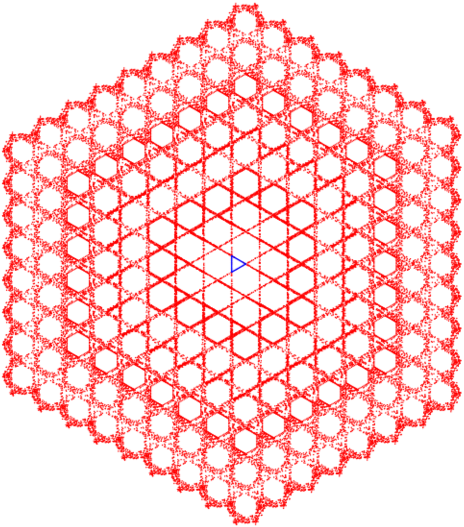
Preserved Region

Theorem

For a PC curve constructed from a unit square with tangent angles $\theta_1, \theta_2, \theta_3, \theta_4$ such that $\theta_1 \leq \theta_3$ and $\theta_2 \leq \theta_4$, preserved region at k-th layer exist if and only if $\tan\theta_1 + \tan\theta_3 \leq \frac{2}{2k-1}$, $\tan\theta_2 + \tan\theta_4 \leq \frac{2}{2k-1}$ *and one of the following is true:*

- $\frac{k\tan\theta_3 - (k-1)\tan\theta_2\tan\theta_3}{1+\tan\theta_2\tan\theta_3} \leq \frac{1}{2}$
- $(k-1)(\tan\theta_3 + 1)\left(\frac{\sin\theta_2}{\cos(\theta_2-\theta_3)} + \frac{\sin\theta_4}{\cos(\theta_3+\theta_4)}\right) \leq \frac{k\cos\theta_3 - \sin\theta_2\sin\theta_3}{\cos\theta_3\cos(\theta_2-\theta_3)}$

Preserved Region



Preserved Region

Proposition

For a regular PC table constructed from a equilateral of unit side length with tangent angle θ, preserved region inside k-th layer of hexagon exist if and only if $\tan\theta \leq \frac{\sqrt{3}}{4k-3}$. If so, the area of the region is given by formula

$$\begin{cases} \frac{3\sin^2(\pi/3-\theta)}{\sin(\pi/3+\theta)\cos\theta} & \text{if } k=1 \text{ and } \theta \leq \pi/6, \\ \frac{3\sin^2(\pi/3-\theta)}{\sin(\pi/3+\theta)\cos(\pi/3-\theta)} & \text{if } k=1 \text{ and } \pi/6 \leq \theta \leq \pi/3 \\ \frac{3((4k-3)\sin(\pi/3+\theta)-(4k-2)\sin(\pi/3)\cos\theta)^2}{\sin(\pi/3+\theta)\cos\theta} & \text{otherwise.} \end{cases}$$

Proposition

For a PC curve constructed from a equilateral of unit side length with tangent angles $\theta_1, \theta_2, \theta_3$, preserved region at k-th layer inside hexagon exists if and only if $\max\{\tan\theta_1, \tan\theta_2, \tan\theta_3\} \leq \frac{\sqrt{3}}{4k-3}$

Preserved Region

Proposition

For a regular PC table constructed from a equilateral of unit side length with tangent angle θ, preserved region inside k-th layer of triangle exist if and only if $\tan\theta \leq \frac{\sqrt{3}}{4k-1}$. If so, the area of the region is given by formula $\frac{\sqrt{3}((4k-1)\sin(\pi/6+\theta)-2k\cos\theta)^2}{2\sin(\pi/6+\theta)\cos\theta}$

Preserved Region

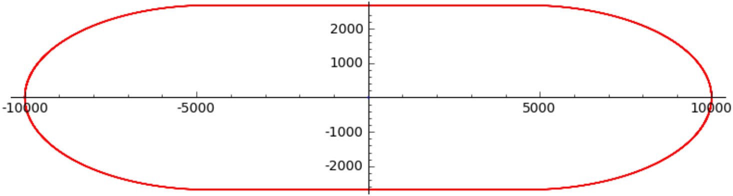
Theorem (Kolodziej)

All orbits in rational polygon are periodic

Open Question

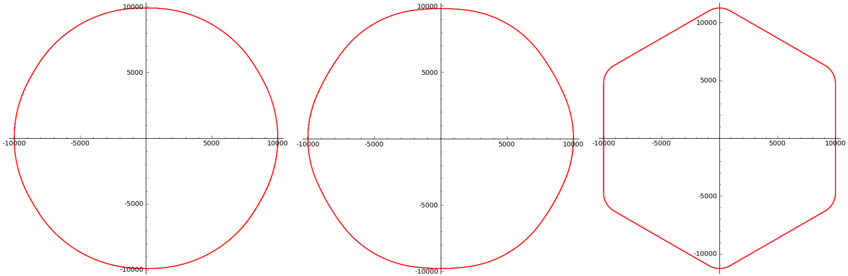
Can we answer similar question about preserved region for any rational polygons? What if we replace sides of polygon by curves that are not piecewise circular?

Vector Field at Infinity



$$(2r\sin\theta, 2\sqrt{r^2-1} - 2r\cos\theta)$$

Vector Field at Infinity



Definition

For a given convex outer billiard table P, a closed curve γ is defined to be an invariant curve of P if $\forall p \in \gamma,\ T(p) \in \gamma$.

Theorem

Given a closed convex curve γ, there exists a one-parameter family of curves η such that γ is an invariant curve for the outer billiard map with respect to η. The curves η are the envelopes of the segments of equal area.
This construction of an outer billiard table η from an invariant curve γ is referred to as the area construction.

Polygonal Invariant Curve

Proposition

Consider two rays r_1, r_2 starting from the same point, the area construction yields a piece of hyperbola.

Proposition

By area construction from a square invariant curve, we get a regular piecewise hyperbola outer billiard table with corners.

Polygonal Invariant Curve

For a regular n-polygons, we define area number $z \in (1, \frac{n}{2})$ as following: we follow the area preserving cutting of the polygon to construct piecewise hyperbola tables. Consider the area being cut off from a segment with one end at a vertex of the polygon: the area A consists of the segment through at least one vertex of the polygon and several sides of the polygon (may contain a part of one side). The area number is defined as the number of sides in A from the original polygon. This number may or may not be an integer or an rational number, and varies from 1 to $\frac{n}{2}$

Proposition

Let $m \in \mathbb{Z}$, $m \geq 2$. Consider a regular (2m+1)-gon as invariant curve, the area preserving construction will result in a piecewise hyperbola table that falls into one of the three categories, depending on the area cut from the original curve (always consider the area that is less than half of the area enclosed by polygon) :
(1) If the area number z is integer, i.e, $z = 2, 3, ...m$, then the construction gives 2m+1 sided regular Ph curve that is C^1 smooth (without corners).

(2) If the area number z is such that $m < z < m + \frac{1}{2}$, then we get 4m+2 sided shape that is not convex and self intersecting.

(3) For all other possible values of z, we get 4m+2 sided PH curves that are not regular but C^1 smooth.

Polygonal Invariant Curve

Proposition

Let $m \in \mathbb{Z}$, $m \geq 3$. Consider a regular (2m)-gon as invariant curve, the area preserving construction will result in a piecewise hyperbola table that falls into one of the three categories, depending on the area cut from the original curve :

(1) If the area number z is integer, i.e, $z = 2, 3, ...m - 1$, then the construction gives 2m sided regular PH curve that is C^1 smooth .

(2) If the area number z is such that $m - 1 < z < m$, then we get 2m sided regular piecewise hyperbola tables that are not C^1 smooth. (meaning it contains corners)

(3) For all other possible values of z, we get 4m sided PH curves that are not regular but C^1 smooth.

Polygonal Invariant Curve

Now we have seen area constructions of C^1 smooth regular piecewise hyperbola tables of both even sides and odd sides (except for a 3 sides PH table). And we have seen constructions of regular PH tables with corners, but only for even sides. Here we present a construction that gives odd sided regular PH table with corners.

Proposition

Consider a regular polygon γ with odd number of sides. We get a new polygon γ' by removing a sufficiently small piece of isosceles triangle from each corner of γ. Then some particular area construction on γ' yields a regular PH table with corners.

Polygonal Invariant Curve

Conjecture

The previous 3 propositions give all possibilities for constructing regular piecewise hyperbola tables.

Polygonal Invariant Curve

Proposition 1

For any C^1 smooth regular piecewise hyperbola table P with 3 sides, there does not exist any polygonal invariant curve for that table P.

Proposition 2

For a 3-sided regular piecewise hyperbola table P, if the tangent line at a vertex has an angle of $\frac{\pi}{6}$ to the side, then the previous proposition gives the only way to construct table P from a polygonal invariant curve.

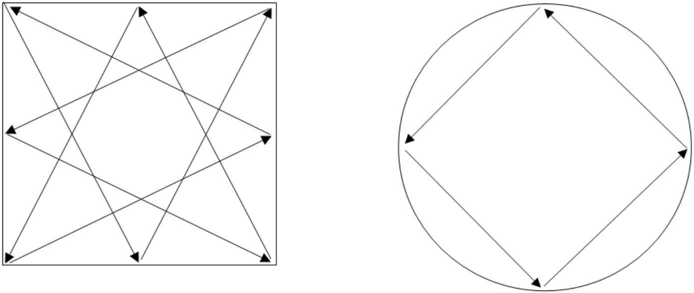


Figure : 8 periodic orbits on square invariant curve [left] and 4 periodic orbits on circular invariant curve [right]

Rotation Number

we consider a circular homeomorphism $f : S^1 = \mathbb{R}/\mathbb{Z} \to S^1$, the natural projection $\pi : \mathbb{R} \to S^1$ provides a lift of the map f to homeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

It is known that F is unique up to adding interger constant.

Rotation Number

Definition (Rotation Number)

The rotation number τ of the map f is defined as:

$$\tau_f = \pi(\lim_{n\to\infty} \frac{F^n(x) - x}{n}).$$

Rotation Number

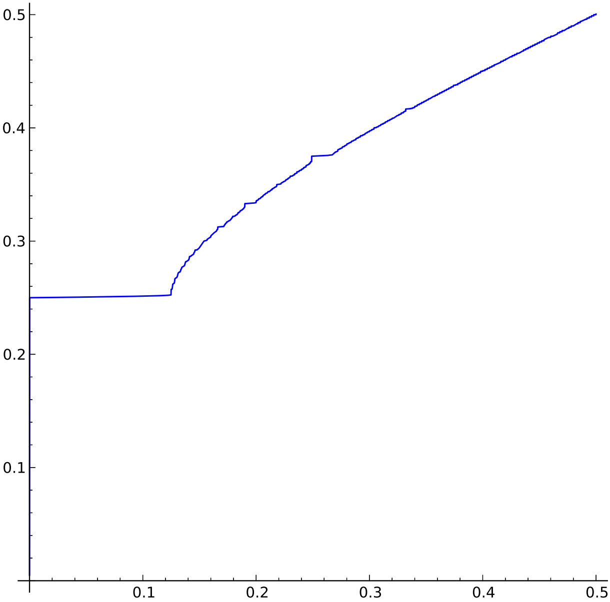
Theorem

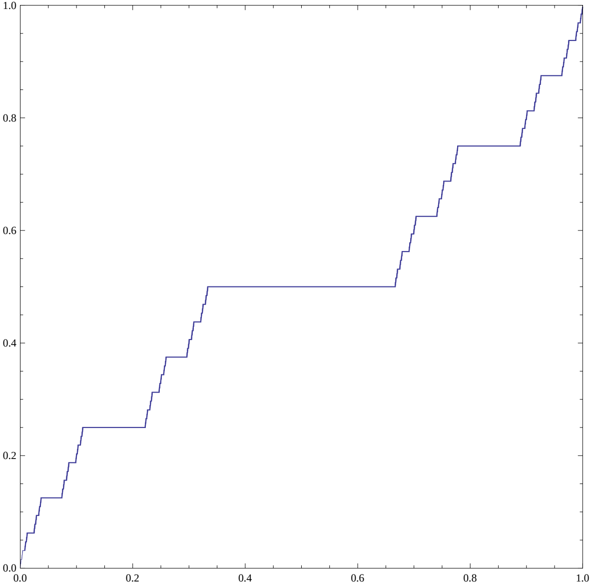
Let $f : S^1 \to S^1$ be circular homeomorphism as discussed above and $F : \mathbb{R} \to \mathbb{R}$ a lift for f, then the limit $\lim_{n\to\infty} \dfrac{F^n(x) - x}{n}$ *exists* $\forall x \in \mathbb{R}$.

Theorem

$\tau_f = \frac{p}{q} \in \mathbb{Q}$ *if and only if f has a periodic orbit of period q (assume $(p, q) = 1$). If this is the case, then every periodic orbit has period q. Furthermore, every forward orbit of f converges to a periodic orbit.*

Rotation Number graph of $\tau(A)$




1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Definition

A continuous and non-decreasing function $\varphi : [0, 1] \to \mathbb{R}$ is called a devil's staircase if there is a family of disjoint open subintervals of $\mathcal{I} = [0, 1]$ such that the union of all these subintervals is dense on $\mathcal{I}$ and the function φ takes distinct **constant** value at each of the subintervals.

Rotation Number

Theorem

$\forall \frac{p}{q} \in \mathbb{Q}$ such that $\frac{1}{4} \leq \frac{p}{q} < \frac{1}{2}$, the set $\tau^{-1}(\frac{p}{q})$ is a non-empty open interval as long as the circular homeomorphism f is not conjugate to a rotation $R_{\frac{p}{q}}$. That is to say, $\tau(a)$ is locally constant at points where the rotation number is rational.

remark: This theorem is a consequence of a more general result about rotation number. An alternative approach was given using geometrical arguments.

Rotation Number

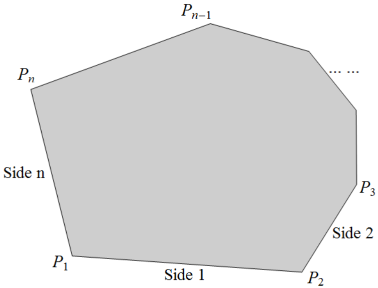
Theorem

For the circular maps $f_a : \gamma \to \gamma$ defined from area constrcution, where γ is the unit square invariant curve. No such f_a is conjugate to a rotation $R_{\frac{p}{q}}$. In fact, if $\tau(a) = \frac{p}{q} \in \mathbb{Q}$, then $f_a{}^q : \gamma \to \gamma$ cannot even be locally identity.

Corollary

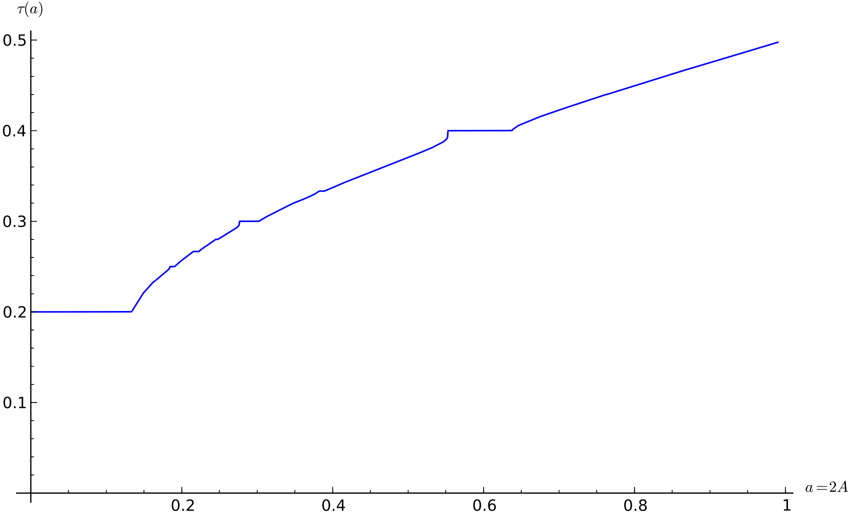
Let $f_a : \gamma \to \gamma$ be circular homeomorphism given by the area construction, this leads to a function of rotation number $\tau(a)$ on parameter $a = 2A$. $\tau(a)$ is a devil's staircase funciton.

Consider a generic convex polygon $\eta = P_1 P_2 ... P_n$.



We follow the area construction to define a one parameter collection of circular maps $f_a : \eta \to \eta$. Again, we study the associated rotation number $\tau(a)$ as a function of $a = 2A$.

Rotation Number



Rotation Number

Theorem

For the circular maps $f_a : \eta \to \eta$ defined from area constrcution, where η is any convex polygon. No such f_a is conjugate to a rotation $R_{\frac{p}{q}}$.
If $\tau(a) = \frac{p}{q} \in \mathbb{Q}$, then $f_a{}^q : \eta \to \eta$ is not identity for any non-empty open interval on η.

Corollary

Let η be a convex polygonal invariant curve and T be a piecewise hyperbola table resulted from the area construction. Then any open interval $I \subset \eta$ contains non-periodic points under the outer billiard map.

Rotation Number

Theorem

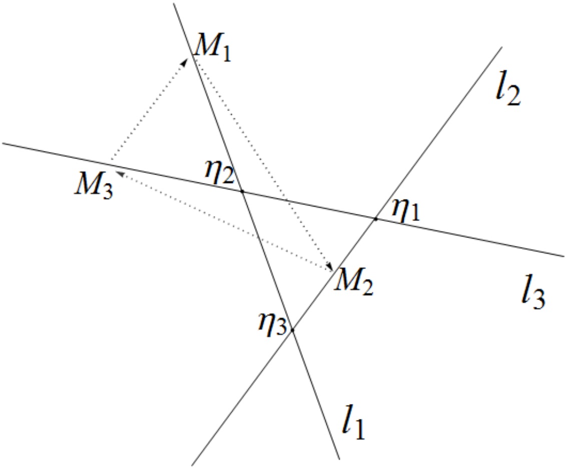
$\tau(a)$ is a devil's staircase funciton for any convex polygon

Rotation Number

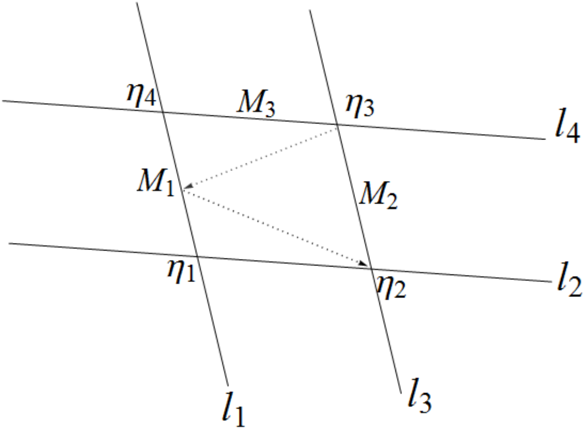
Given an ordered collection of lines $l_1, l_2, ..., l_n$, not necessarily distinct, we define a series of functions $f_k : l_k \to l_{k+1}, \quad k = 1, ..., n, \quad$. Each function is given by the associated area construction map between two consecutive lines l_k, l_{k+1}.

We ask the following questions, does there exist such a collection of ordered lines with the defined maps $f_1, ..., f_n$ such that $f_n \circ f_{n-1} \circ ... \circ f_1 = identity$?

Rotation Number



Rotation Number



Rotation Number

From the examples of the three ordered lines, we could construct a non-simple polygon (A polygon that crossses itself) such that there exists a local open interval of periodic points under some composed map defined analogously.

This tells us that our theorem cannot be generalized to arbitrary collection of lines or arbitrary polygon.

Now, what about a non-simple and non-convex polygon? We define the circular maps from polygon to itself by area construction, could there exist an open interval on which some qth power of the map gives identity?

1. Does there exist a curve such that all points are periodic under some qth power of the ciruclar map defined previously?

2. What about a simple and non-convex polygon? Could there exist an open interval on which some qth power of the map gives identity?

3. Is the answer to the previous question happens to be "yes", could this be a global identity ?

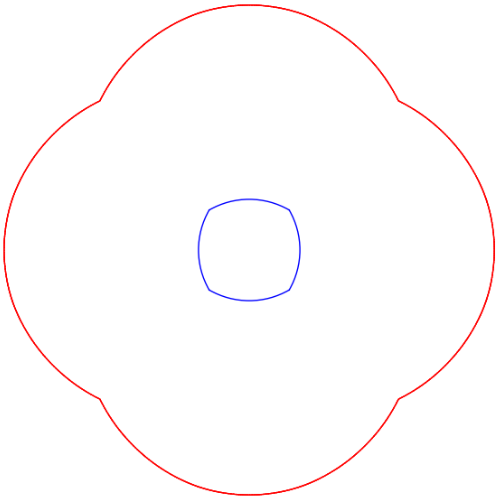
4. What about other shapes of invariant curve? For example, piecewise conic invariant curves?

Future Work

- Consider outer billiard map of non-strictly convex table
- Existence of Invariant curve about piecewise circular curve
- Chaotic behavior of outer piecewise hyperbola curve

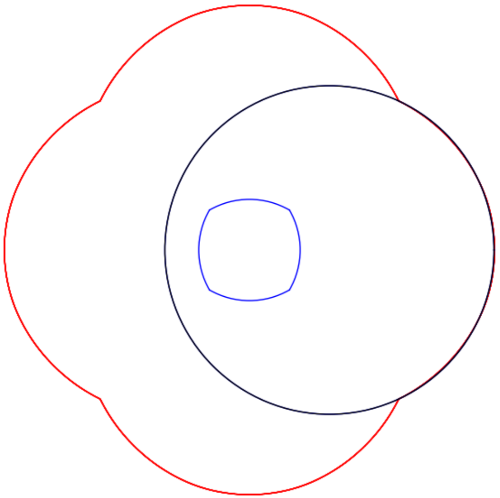
Future Work

Existence of Invariant curve about piecewise circular curve



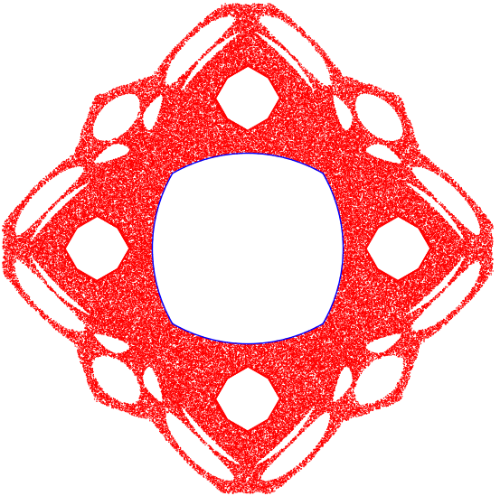
Future Work

Existence of Invariant curve about piecewise circular curve



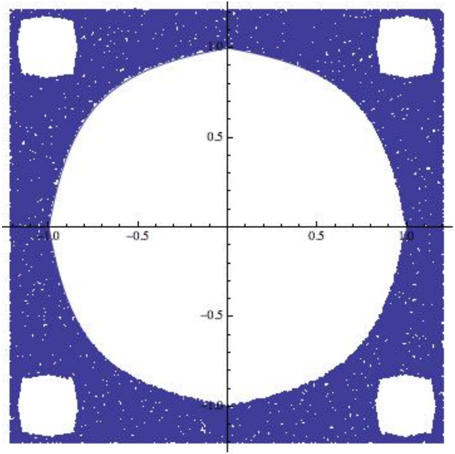
Future Work

Existence of Invariant curve about piecewise circular curve



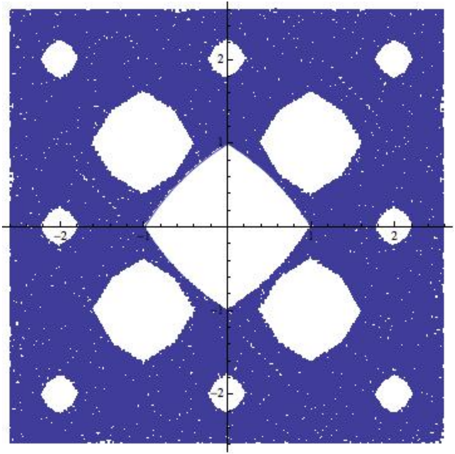
Future Work

Chaotic behavior of outer piecewise hyperbola curve



Future Work

Chaotic behavior of outer piecewise hyperbola curve



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