# Research projects

December 11, 2012

## 1 Billiards and related systems

### 1.1 Polygonal outer billiards in the hyperbolic plane

The outer billiard about a convex polygon $P$ in the plane $\mathbf{R}^2$ is a piece-wise isometry, $T$, of the exterior of $P$ defined as follows: given a point $x$ outside of $P$, find the support line to $P$ through $x$ having $P$ on the left, and define $T(x)$ to be the reflection of $x$ in the support vertex. See [10, 35].

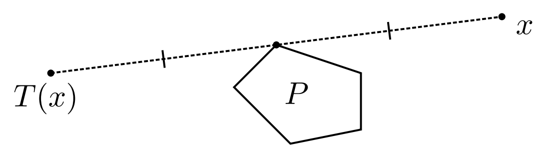



Figure 1: The outer billiard map in the plane

C. Culter proved (Penn State REU 2004) that every polygon in the plane admits periodic outer billiard orbits, see [36]. Outer billiard can be defined on the sphere and in the hyperbolic plane. On the sphere, there exist polygons without periodic outer billiard orbits. **Conjecture**: *every polygonal outer billiard in the hyperbolic plane has periodic orbits.* These orbits may lie on the circle at infinity.

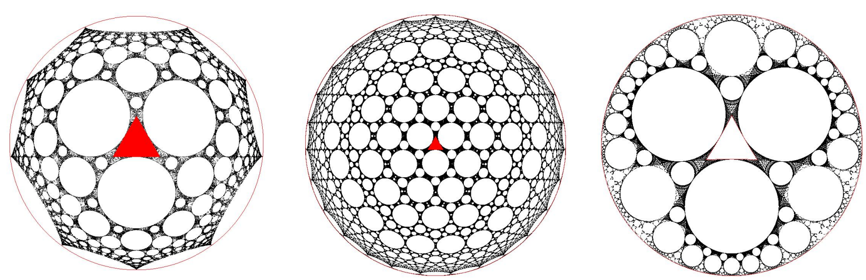


Figure 2: Outer billiards on equilateral triangles in the hyperbolic plane

Figure 2 illustrates the complexity of this system, even in the case of an equilateral triangle: the white discs are periodic domains of the outer billiard map.

Another interesting problem is *to describe polygonal outer billiard tables in the hyperbolic plane for which all orbits are periodic.* For example, right-angled regular $n$-gons (with $n \geq 5$) have this property, see [9]. In the affine plane, every outer billiard orbit about a lattice polygon is periodic.

## 1.2 Polyhedral outer billiards in 4-dimensional space

Let $M$ be a closed convex hypersurface in $\mathbf{C}^2$. For a point $x \in M$, let $n(x)$ be the outer unit normal vector. The *outer billiard map* $T$ of the exterior of $M$ is defined as follows. For $t > 0$, consider points $y = x + itn(x)$ and $z = x - itn(x)$ (of course, $i = \sqrt{-1}$); then $T(y) = z$. One can prove that for every $y$ outside of $M$ there exists a unique $x \in M$ and $t > 0$ such that $y = x + itn(x)$, hence the map $T$ is well defined. See [10, 35] for more details.

**Problem**: *study the dynamics of the outer billiard map when $M$ is the surface of a regular polyhedron in $\mathbf{C}^2$.*

In dimension four, there are six regular polyhedra. Even for a regular simplex, one expects an interesting dynamical system.

In the plane, a regular pentagon (and other regular $n$-gons with $n \neq 3, 4, 6$) yields a beautiful fractal set, the closure of an infinite orbit of the outer billiard map. See Figure 3.

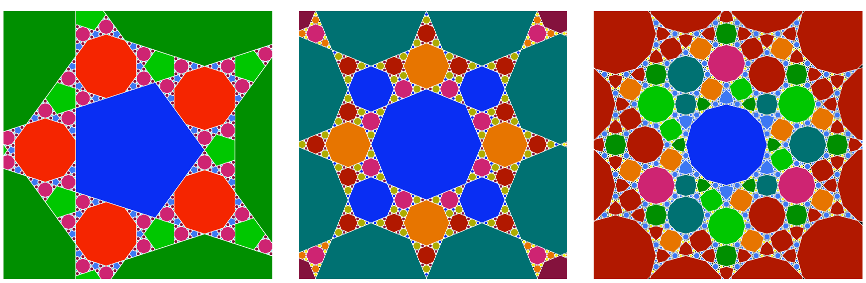


Figure 3: Outer billiards on regular $n$-gons: $n = 5, 8, 12$

## 1.3 Outer billiards about piecewise circular curves

An interesting class of curves to study as outer billiard tables are piecewise circular curves. For such curves, the outer billiard map is continuous (although not everywhere differentiable). One asks the usual questions: *are there periodic orbits? can orbits escape to infinity? can orbits fall in the outer billiard table?* and so on. See [1] for the geometry of piecewise circular curves.

## 1.4 Magnetic billiards

Magnetic billiards describe free motion of a charge in magnetic field with elastic reflection of the boundary of a plane domain (a billiard table). If the magnetic field is constant then the charge moves along an arc of a circle of a fixed radius. See [3] or [34].

Not much is known about magnetic billiards. An interesting problem is *to study magnetic billiard inside a square in constant magnetic field.* One can unfold a trajectory to a pice-wise circular curve (similarly to unfolding a billiard trajectory to a straight line). Will these curves be unbounded? Will they have limiting directions?

One can ask the same question about other polygons that tile by reflection (such as the $30^\circ, 60^\circ, 90^\circ$ triangle).

## 1.5 Negative Snell law

*Snell's Law* describes what happens to a beam of light when it passes from one medium to another – say, from air to glass. The angles of incidence and refraction are related to the refractive indices of the two media. Recently, physicists have discovered materials that have *negative* indices of refraction.

Consider a two-colorable tiling where the tiles are fabricated out of materials with opposite indices of refraction. If we shine a beam of light into the tiling, then when it crosses a boundary between tiles, it "bounces back" with an angle of reflection equal to the angle of incidence. An example is below. (In fact, we will generally ignore the requirement that the tiling is two-colorable.)

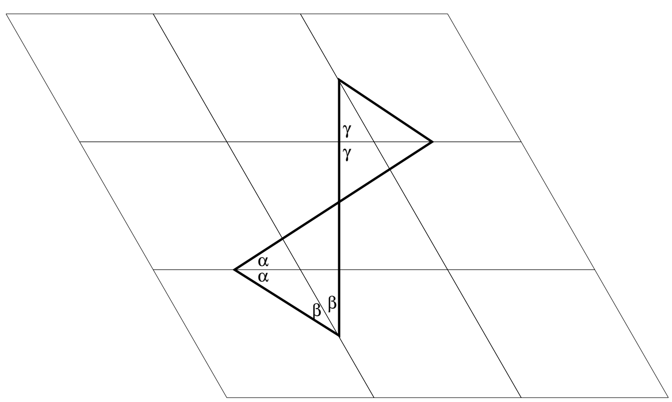



Figure 4: A periodic path with period 8 and angle $\pi/3$

A recent paper [26] investigated some of the questions associated to these tilings. For the square grid tiling, there are only two possibilities: the path forms a period-4 rhombus, or an infinite horizontal or vertical saw-tooth pattern.

For the equilateral triangle tiling, every path circles a vertex periodically, with period 6.

At Summer@ICERM 2012, two students studied this system and got some results about its local behavior. Since then, our TA Diana Davis has been studying parallelogram tilings like the one above, for various length and angle parameters.

Still, most of the basic questions about dynamical systems remain open for this system, and we will work on resolving them.

### 1.6 Spherical and hyperbolic versions of Gutkin's theorem

E. Gutkin asked the following question: given a plane oval $\gamma$, assume that two points, $x$ and $y$, can "chase" each other around $\gamma$ in such a way that the angle made by the chord $xy$ with $\gamma$ at both end points has a constant value, say, $\alpha$. If $\gamma$ is not a circle, what are possible values of $\alpha$?

The answer is as follows: a necessary and sufficient condition is that there exists $n \geq 2$ such that $n \tan \alpha = \tan(n\alpha)$. See [23, 32].

In terms of billiards, the billiard ball map in $\gamma$ has an invariant circle given by the condition that the angle made by the trajectories with the boundary of the table is equal to $\alpha$. The result can be also interpreted in terms of capillary floating with zero gravity in neutral equilibrium, see [12, 13].

**Problem**: *find analogs of this result in the spherical and hyperbolic geometries. What about curves in higher dimensional spaces?*

## 2 Geometry

### 2.1 The modeling of constant curvature surfaces in space

Find a surface of constant negative curvature in space.

This does not seem like a remarkably difficult problem: for example, we know quite a lot about the intrinsic properties of curvature, and have the powerful Hilbert Embedding Theorem, which at least places heavy extrinsic constraints on constant negative curvature surfaces in space. Negatively curved surfaces are ubiquitous in nature, and there are several nice methods for making models of constant negative curvature surfaces out of paper, yarn or steel [37].

The first sign of trouble, perhaps, is that very few constant negative curvature surfaces appear to be explicitly described, say through a parametric equation—this author knows of none discovered in the last 125 years.

The problem here is to generate good computer approximations to such a surface, by "evolving" discrete models. Though these techniques have been used with spectacular results for minimal surfaces, there are several numeric, computational and mathematical subtleties that come into play in this setting.

Two approaches are proposed: The first is to mimic the assembly of straight strips of paper outlining a surface, much like the sculpture shown in Figure 5. The lengths of the strips, and the angles they meet at, are fixed, but their

Figure 5: A surface of constant negative curvature

arrangement in space is not; additionally one would like to minimize the total amount of "bending energy", that is, the strips are viewed as trying to lie as straight and flat as possible. This approach guarantees that curvature is constant at all steps, but appears to be computationally ill-posed.

The second is to consider a random mesh in the hyperbolic plane; each edge of the mesh has a certain length. In space, a combinatorially equivalent mesh is evolved, through simulated annealing, driving its edge lengths to the corresponding lengths in the hyperbolic mesh. One open experimental question is: As the edges approach the correct lengths, does the discrete curvature converge uniformly across the surface?

(This problem arose in a very applied setting— the design of an exhibit on curvature for the new Museum of Mathematics.)

## 2.2 Origami hyperbolic paraboloid

There is a common origami construction depicted in Figure 6. The pleated surface looks like a hyperbolic paraboloid and is often called so in the origami literature.

The problem is to explain this construction. If one assumes that paper is not stretchable and the fold lines are straight then one can prove that this construction is mathematically impossible, see [7]. The explanation is that there exist invisible folds along the diagonals of the elementary trapezoids in Figure 6 left. Assuming this to hold, and given a particular patterns of these diagonals (one can choose one of the two for each trapezoid), *what is the shape of the*

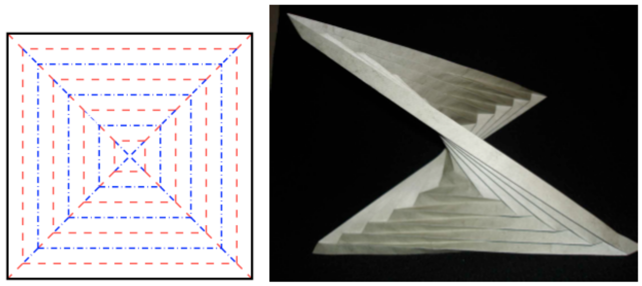


Figure 6: Hyperbolic paraboloid

*piece-wise linear surface obtained by folding?*

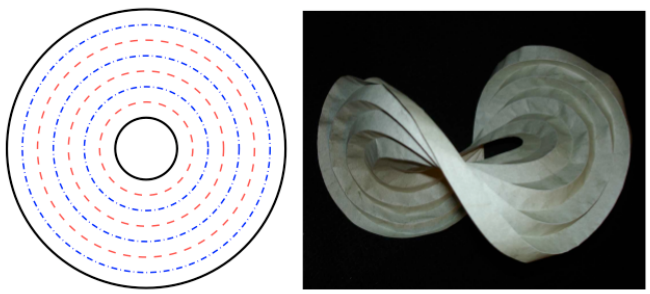


Figure 7: A circular pleated surface

A more general question: *what is the result of a similar construction for other patterns of folding lines?* See, e.g., Figure 7. See [14] concerning folding paper along curved lines and the books [8, 29] for mathematical paper folding.

### 2.3 The unicycle problem and its ramifications

A mathematical model of a bicycle is an oriented unit segment $AB$ in the plane that can move in such a way that the trajectory of the rear end $A$ is always tangent to the segment. Sometimes the trajectories of points $A$ and $B$ coincide (say, riding along a straight line).

The following construction is due to D. Finn [11]. Let $\gamma(t),\ t \in [0, L]$ be an arc length parameterized smooth curve in the plane which coincides with all derivatives, for $t = 0$ and $t = L$, with the $x$-axis at points $(0, 0)$ and $(1, 0)$, respectively. One uses $\gamma$ as a "seed" trajectory of the rear wheel of a bicycle. Then the new curve $\Gamma = T(\gamma) = \gamma + \gamma'$ is also tangent to the horizontal axis with all derivatives at its end points $(1, 0)$ and $(2, 0)$. One can iterate this procedure yielding a smooth infinite forward bicycle trajectory $\mathcal{T}$ such that the tracks of the rear and the front wheels coincide. See Figure 8.

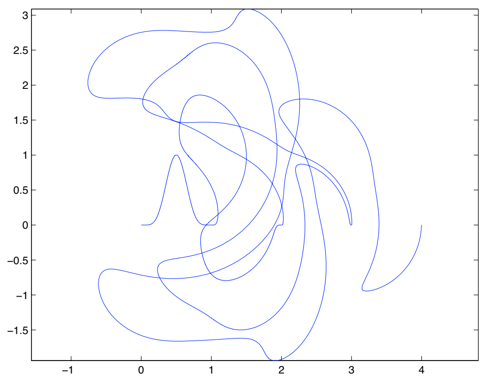



Figure 8: A unicycle track

It is proved in [25] that the number of intersections of each next arc of $\mathcal{T}$ with the $x$-axis is greater than that of the previous one. Likewise, the number of local maxima and minima of the height function $y$ increases with each step of the construction. As a result of Summer@ICERM 2012 research [28], we also know that the number of inflection points increases with each step of the construction.

**Conjecture**: *Unless $\gamma$ is a straight segment, the amplitude of the curve $\mathcal{T}$ is unbounded, i.e., $\mathcal{T}$ is not contained in any horizontal strip, $\mathcal{T}$ is not a graph, and $\mathcal{T}$ is not embedded, that is, it starts to intersect itself.*

Here is a related problem. Given an oriented oval $\gamma$, draw the unit tangent segments to $\gamma$, and let $\gamma_1$ be the locus of their endpoints. We get a map $\gamma \mapsto \gamma_1$.

**Conjecture**: *If all iterations of this map are convex curves then $\gamma$ is a circle*, see Figure 9.

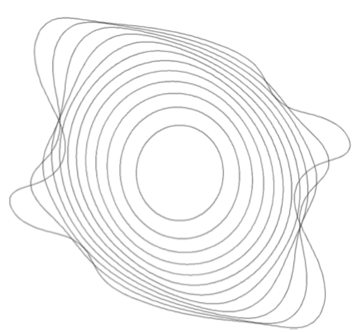


Figure 9: Convexity is fragile

A justification is a linearization of this statement, which is a theorem: *Let $F$*

*be a periodic function, and consider the linear map $F \mapsto F + F'$. If all iterations of $F$ are positive then $V$ is a positive constant.*

### 2.4 Tripod configurations

A tripod configuration for a plane curve $\gamma$ is a triple of points on the curve such that the normals to $\gamma$ at these points are concurrent and make the angles of $120^\circ$ with each other. It is proved in [33] that each plane oval has at least two tripods. *Extend this result to non-convex and to self-intersecting curves. What about closed curves in 3-space: do they necessarily admit tripod configurations?*

A possible approach to the problem is variational: a tripod $ABCD$, where points $A, B, C$ are on the curve and $D$ is the intersection point of the normals making the angles of $120^\circ$, is a critical point for the function $|AD|+|BD|+|CD|$ on quadruples of points $ABCD$ with the constraint $A, B, C \in \gamma$.

### 2.5 Self-dual curves and surfaces

Projective duality is a correspondence between points of the real projective plane $\mathbf{RP}^2$ and lines of the dual projective plane $(\mathbf{RP}^2)^*$; projective duality extends to smooth and piece-wise smooth curves, taking a curve $\gamma \subset \mathbf{RP}^2$ (a one-parameter family of points) to the envelope $\gamma^* \subset (\mathbf{RP}^2)^*$ of the respective one parameter family of dual lines. A curve $\gamma$ is called projectively self-dual if there exists a projective transformation from $\mathbf{RP}^2$ to $(\mathbf{RP}^2)^*$ that takes $\gamma$ to $\gamma^*$. Likewise one defines self-dual polygons.

The general problem of describing projectively self-dual curves is poorly understood, see [15] for a description of projectively self-dual polygons and some results on self-dual curves.

**Problem**: *extend the results of [15] to projectively self-dual hypersurfaces and polyhedra, and to projectively self-dual non-degenerate curves and polygons in multi-dimensional projective spaces.*

A non-degenerate curve $\gamma$ in $\mathbf{RP}^n$ has the osculating hyperplane at each point; a hyperplane in $\mathbf{RP}^n$ is a point in the dual projective space $(\mathbf{RP}^n)^*$, and this one-parameter family of points in $(\mathbf{RP}^n)^*$ is the dual curve $\gamma^*$.

All these problems have affine analogs in which the curves (hypersurfaces) are assumed to be star-shaped and the projective duality is replaced by the polar duality.

### 2.6 Cayley-style theorem for null geodesics on an ellipsoid in Minkowski space and for the zig-zag theorem

The following Poncelet-style theorem was proved in [16]. Consider an ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1, \quad a, b, c > 0$$

in three dimensional Minkowski space with the metric $dx^2 + dy^2 - dz^2$. The induced metric on the ellipsoid degenerates along the two "tropics"

$$z = \pm c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}};$$

the metric is Lorentz, of signature $(+, -)$, in the "equatorial belt" bounded by the tropics. Through every point of the equatorial belt there pass two null geodesics of the Lorentz metric, the "right" and the "left" ones.

Call a chain of alternating left and right null geodesics, going from tropic to tropic, an $(n, r)$-chain if it closes up after $n$ steps and making $r$ turns around the equator. The theorem states that if there exists an $(n, r)$-chain of null geodesics then every chain of null geodesics is an $(n, r)$-chain. See [16] for a discussion.

**Problem**: *find conditions on the numbers $a, b, c$ ensuring the existence of $(n, r)$-chains.*

For the classical Poncelet porism, such a formula is due to Cayley, see [21]. For the Poncelet porism, see [2] and [6].

A similar question can be asked about the zig-zag theorem [2, 4] concerning two circles in Euclidean 3-space, positioned in such a way that, for some number $d$, each point of either circle is distance $d$ from exactly two points of the other circle (this is not too restrictive). Consider a chain of points $x_1, x_2, \dots$ so that even points lie on one circle, odd points on another circle and $|x_i - x_{i+1}| = d$ for all $i$. The claim is that if one such chain closes up after $n$ steps then so does every such chain.

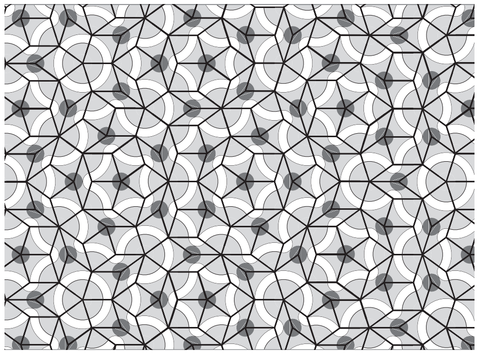
**Problem**: *find conditions on the circles and $d$ ensuring that the chain closes after $n$ steps making $k$ turns around the circles.*

Another problem is to generalize the zig-zag theorem to two circles in Euclidean spaces of dimension 4 and 5, and to the spherical or hyperbolic spaces.

# 3 Quasi-periodic structures

## 3.1 Cut-and-project tiling in the hyperbolic plane

Throughout the 1980's the "cut-and-project" method was widely studied as a method for understanding conjectured structures of quasicrystals, physical materials with apparently forbidden crystallographic structure, discovered in 1982 by Daniel Schechtman [30]. In Figure 10 (left) of the famous Penrose tiles, the rhombuses seem to form squashed cubes— and indeed they are the projections of faces in $\mathbb{Z}^5$, those closest to a particular plane cutting through this lattice. More precisely, we might typically take two non-trivial subspaces $V_1$ and $V_2$ of $\mathbb{R}^n$ such that $V_1 \oplus V_2 = \mathbb{R}^n$; letting $\pi_1$ be the projection of $\mathbb{R}^n$ to $V_1$ with kernel $V_2$ and $\pi_2$ be the projection of $\mathbb{R}^n$ to $V_2$ with kernel $V_1$, suppose $\pi_1(\mathbb{Z}^n)$ is one-to-one and $\pi_2(\mathbb{Z}^n)$ is dense in $V_2$. Let $\Omega$ (the "acceptance window") be a compact set with non-empty interior; then a "cut-and-project set" $\{\pi_1(x) \mid x \in \mathbb{Z}^n, \pi_2(x) \in \Omega\} \subset V_1$. A simple example, of a cut and project set in the line, derived from $\mathbb{R}^2$, is shown in Figure 10, right.



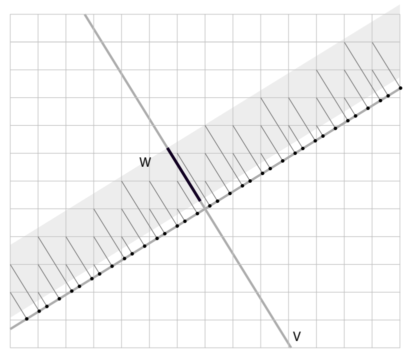



Figure 10: (Left) Tilings by the Penrose tiles appear to be formed from squashed cubes—and indeed they are; the tiles are naturally projected from $\mathbb{Z}^5$ to $\mathbb{R}^2$ via the cut-and-project method. (Right) A simple example of a cut-and-project set in the line; notice the interesting pattern of "long" and "short" intervals: `...lslsllslsllsllslsllsllslsllslsllsllsl...` which is highly regular, yet is not periodic.

The method has been extensively explored in Euclidean space, but not elsewhere.

Samuel Petite has proposed an approach to constructing cut-and-project tilings in the hyperbolic plane; this project is to work out the details and produce actual tilings, almost certainly stimulating further research. His approach is to construct a co-cocompact, irreducible lattice within $PSL(2, \mathbb{R})^2$, the group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$. That is, of course, the mathematical heart of the problem; once this is in hand, this lattice induces an action on $\mathbb{H}^2 \times \mathbb{H}^2$; some orbit is chosen, and the points within some fixed distance of $0 \times \mathbb{H}^2$ are cut and projected to the hyperbolic plane. The choice of orbit and window may provide interesting questions.

In particular, many Euclidean cut-and-project tilings exhibit a self-similar structure–exact conditions for this have been extensively studied. In the hyperbolic plane, no self-similarity is possible. On the other hand, as discussed in the next section, a rich variety of highly organized, yet non-periodic structures are possible—does this cut-and-project method correspond to these?

## 3.2 Aperiodic tiles in other geometries

Aperiodic sets of tiles are those that can form tilings, but cannot form any periodic tiling. It is quite remarkable, really, that this is possible at all—somehow translational order has to be disrupted at all scales.

The most famous of these sets is surely the Penrose tiles, shown in Figure 10 [22]. In the Euclidean plane, higher dimensional Euclidean spaces, the hyperbolic plane, and other settings quite a lot is known. (One survey on aperiodic tile sets and related issues appears at [20].)

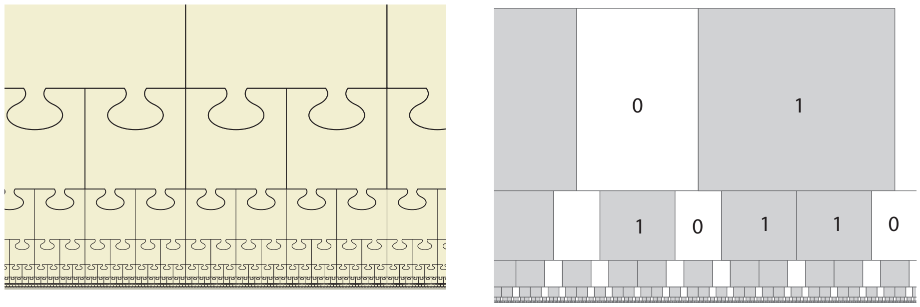



Figure 11: Tilings by two weakly aperiodic sets of tiles in $\mathbb{H}^2$. On the left, a single aperiodic "square" with horocyclic top and bottom—Can you supply the proof that there can be no compact fundamental domain for a tiling by these tiles? At right, tiles corresponding to the "Fibonacci" symbolic substitution system, in which strings of `0`'s and `1`'s have each `0` replaced with `10` and each `1` replaced with `110` (corresponding to one string of `0` and `1` tiles sitting on top of the replacement string of tiles. Note that the pattern of `0`'s and `1`'s is the same as the pattern of `l`'s and `s`'s in Figure 10.

We distinguish "weakly" aperiodic sets of tiles —those that form tilings but cannot form any tiling with a co-compact symmetry— and "strongly" aperiodic sets of tiles, which can only form tilings with no infinitely cyclic symmetry whatsoever. In hyperbolic space, at least, it is almost trivial to construct weakly aperiodic sets—the tiles shown in Figure 11 are example. It is much more subtle to find strongly aperiodic ones [18].

In 1992 Block and Weinberger gave a general, but relatively abstract, construction for weakly aperiodic sets of tiles in a wide range of "non-amenable" spaces [5], including $\widetilde{PSL}(2,\mathbb{R})$. S. Mozes gave another construction in 1997.[27]

The first question here is to produce an explicit description of a weakly aperiodic sets of tiles in $\widetilde{PSL}(2,\mathbb{R})$, and then leverage the constructions of strongly aperiodic sets of tiles in $\mathbb{H}^2$ to produce the first explicit construction of strongly aperiodic sets of tiles in $\widetilde{PSL}(2,\mathbb{R})$. The project may be continued, using similar techniques to produce strongly aperiodic sets of tiles in other model geometries.

### 3.3 Orbits of symbolic substitutions and points at infinity for tilings

At right in Figure 11, we see a tiling of $\mathbb{H}^2$; infinite rows of tiles correspond to infinite strings of `0`'s and `1`'s, and one row sits atop another if they are related by a particular symbolic substitution, $0 \mapsto 10$, $1 \mapsto 110$. In fact, the entire tiling corresponds, precisely to an orbit, under this substitution, in the space of all infinite strings of `0`'s and `1`'s. It is not difficult to show that, even up to "reindexing" there are uncountably many distinct orbits (all of which look exactly the same in any finite region) corresponding to uncountably many

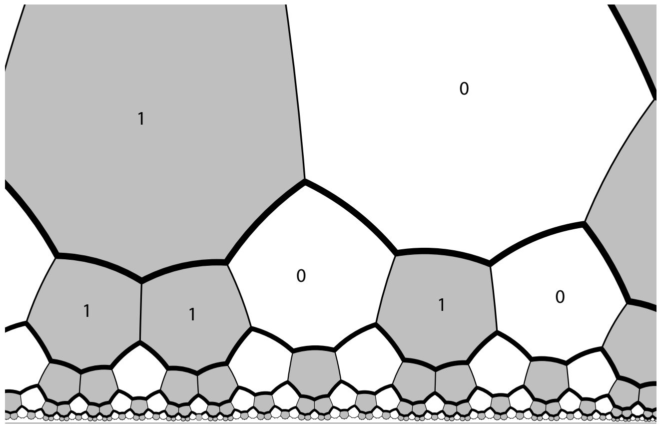



Figure 12: (Compare to the right of Figure 11) The $\{7,3\}$ tiling has been partitioned into strips corresponding to an orbit under the Fibonacci symbolic substitution system. Does every point at infinity uniquely correspond to such a partition?

distinct tilings, up to congruence, by these tiles (ditto). The paper [19] contains many definitions, examples, problems and applications.

Now consider a regular tiling of the hyperbolic plane, say, $\{7,3\}$, the tiling by heptagons meeting three at a vertex. As shown in Figure 3.3, we may consider such a tiling as stacked ribbons, one tile thick, all converging on a single point at infinity. Each tile is oriented in one of two ways within a ribbon, which we label `0` or `1`. Remarkably, each ribbon is related to the next by precisely the substitution system described above, and, up to congruence, each division of the heptagonal tiling corresponds, up to congruence, precisely to a tiling by the tiles of Figure 11 and, up to reindexing, an orbit in the symbolic substitution system.

It's clear that such a division of a regular tiling must correspond to a particular point at infinity: as the strips are all of bounded distance apart, they must all converge at a particular point. But the converse is not clear:

Fix a tiling of $\mathbb{H}^2$ (say $\{7,3\}$) Given a point at infinity, is there a division of the tiling into strips one tile across so that the strips all limit to this point? Is this division unique?

The problem would settle the precise relationship between these substitution systems and tilings of $\mathbb{H}^2$, within the broader context outlined in [19].

## 3.4 Quasiperiodic polyhedral foams

"Dodecafoam" is a self-similar tiling of space by dodecahedra, described more precisely in [17], which appears to be closely related to the conjectured structure of certain "quasicrystals"[31]. Dodecafoam is defined through a particular substitution on tiles in space, but in the end satisfies five local rules:

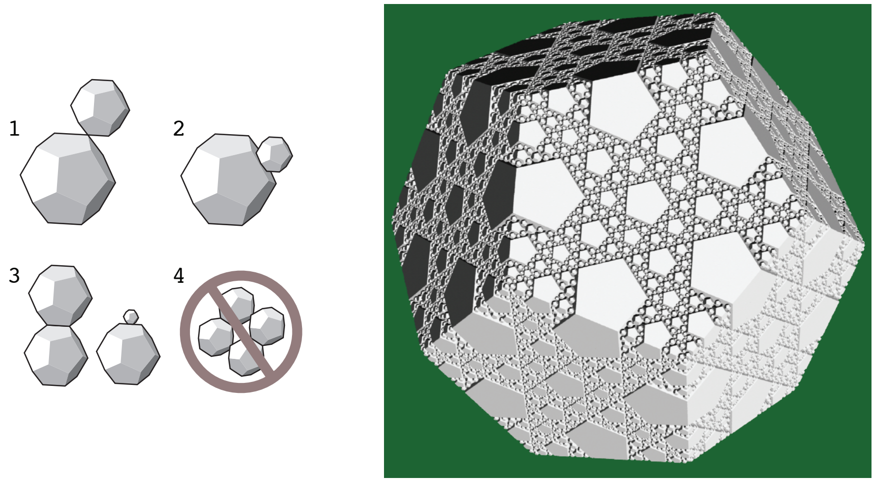



Figure 13: Dodecafoam satisfies five local rules.

1. Dodecahedra meeting vertex to vertex are scaled to one another by $-\phi^1 = -\frac{1+\sqrt{5}}{2}$ through this vertex.
2. The center of each face of a dodecahedron meets the center of the face of another, scaled by $-\phi^2$ through this center.
3. The center of each edge of a dodecahedron meets the center of the edge of another, scaled by $-\phi^{3 \text{ or } 0}$ through this center.
4. There are no chains of 4 dodecahedra meeting edge to edge.

and of course

0. The dodecahedra have disjoint interiors and the closure of their union is the entire space.

The first question is whether or not any tiling of space by dodecahedra that satisfies these rules necessarily non-periodic? Necessarily self-similar? Necessarily the same precise recursive structure as dodecafoam?

If the answer to this last is “yes”, then nicely stated local matching rules have been produced, local rules that guarantee the accurate assembly of tiles into this particular globally defined structure. If the answer to the first is yes and the third is “no”, then a larger, more loosely structured space of tilings is available to explore. Finally, are there other interesting foams to be found.

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