SUMMER@ICERM 2014

DESCRIPTION OF PROBLEMS

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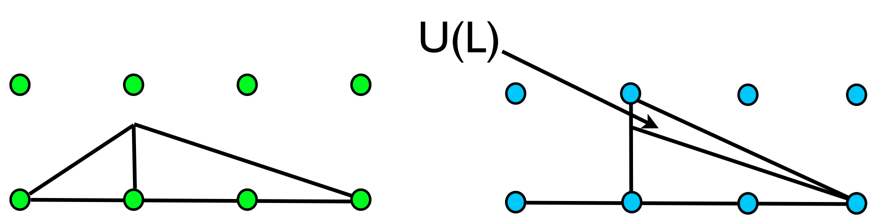
We describe a number of interesting, open-ended problems in combinatorics, discrete geometry, and number theory for study at the Summer@ICERM 2014 program. These problems arise in current research, are amenable to computation and experimentation, and may be approached from a number of different angles. A number of references are listed for each problem.

Periods of rational polygons. A *rational polygon* $\mathcal{P}$ is a polygon in the plane, all of whose vertices have rational coordinates. If we dilate the polygon $\mathcal{P}$ by an integer k, the dilated polygon is written as $k\mathcal{P}$. It is a theorem that the number of integer points in the plane $(a,b) \in \mathbb{Z}^2$, which are contained in $k\mathcal{P}$, is given by a quasi-polynomial function of k. That is, $|\mathbb{Z}^2 \cap k\mathcal{P}| = \text{area}(\mathcal{P})k^2 + a_1(k)k + a_0(k)$, where $a_1(k)$ and $a_0(k)$ are periodic functions of k.

The study of the periods of the number-theoretic functions $a_1(k)$ and $a_2(k)$ is an ongoing research topic, but very little is known about them. We can experiment computationally with many concrete examples, obtain conjectures about these periods, and then confirm and refine them. The known theory, sometimes called *Ehrhart theory of rational polytopes*, extends naturally to higher d-dimensional polytopes, where we get d-dimensional quasi-polynomials. Of course, computations there become more challenging, although quite tractable up to, say, dimension $d = 12$. Among the various applications are bounds on the Frobenius coin exchange problem, and discrete approximations to the volume of a polytope, which we call discrete volumes.

The analysis of the periods may be tackled by using pure combinatorics, or by using generating functions and discrete Fourier analysis. In some recent work by Hasse and McAllister on "period collapse," the authors point out a fascinating conjecture telling us that if we have a period of 1 for a rational polygon that is 'supposed' to have a priori a larger period, then we can dissect it into smaller rational polygons, put them together again by using orthogonal transformations, and find an integer polygon with integer vertices having the same discrete volume. This is, hence, a kind of discrete form of Hilbert's third problem, on equidecomposability. Some related work has been carried out by the second author.



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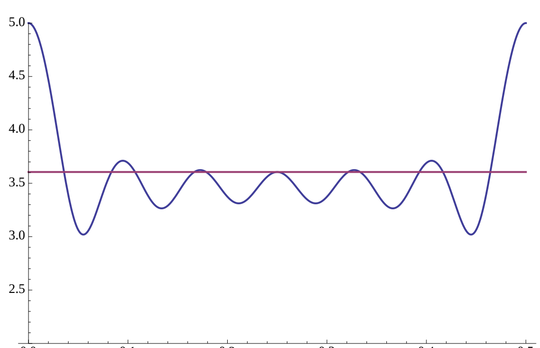
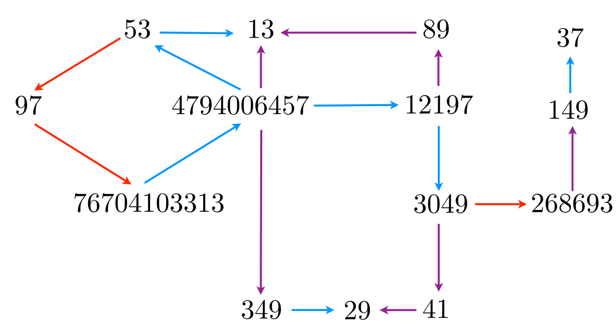
Barker sequences. A *Barker sequence* is a finite sequence $a_1, \ldots, a_n$, each term ± 1, with the property that each of its aperiodic autocorrelations $c_k = \sum_{i=1}^{n} a_i a_{i+k}$ with $k > 0$ is 0, 1, or -1. It is widely conjectured that there are no Barker sequences with length greater than 13, and this problem remains open. It is a special case of the well-known circulant Hadamard matrix conjecture, which asserts that if an $n \times n$ circulant matrix with all entries ± 1 has mutually orthogonal columns, then $n \leq 4$. A number of required conditions are known for n in these problems, so many in fact that it was determined only recently that there exist positive integers $n > 13$ that cannot be disqualified as the length of a Barker sequence. The first author, with P. Borwein, recently determined that the smallest such integer is $n = 3\,979\,201\,339\,721\,749\,133\,016\,171\,583\,224\,100$. This problem is also related to several interesting questions regarding polynomials. For example, it is known that a long Barker sequence gives rise in a natural way to a polynomial that is unusually flat over the unit circle.

There are many avenues of future computational explorations available here, including improving methods for finding exceptional integers in the Barker sequence or circulant Hadamard matrix problem (this involves searches for Wieferich prime pairs (q, p), which satisfy $q^{p-1} \equiv 1 \bmod p^2$), and investigating variations of this problem, where one allows more freedom in the values of the autocorrelations. For example, can one find an infinite family of binary sequences where all of the off-peak aperiodic autocorrelations are small in some sense, say bounded by $\sqrt{n}$?





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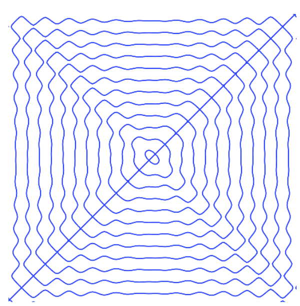
Visualizing harmonic analysis: the zero set of the Fourier transform of a polygon. We propose a visual study of the harmonic analysis of rational polygons. When we compute the (continuous) Fourier transform of a rational polygon $\mathcal{P}$, we obtain a function of two complex variables, whose set of real zeros in $\mathbb{R}^2$ tells us whether $\mathcal{P}$ (multiply) tiles the plane or not. By definition, we say that $\mathcal{P}$ multiply tiles the plane by a discrete set of translations if each point of $\mathbb{R}^2$ is covered exactly the same number of times by some finite set of translates of $\mathcal{P}$. When $\mathcal{P}$ is symmetric about the origin (so $\mathbf{x} \in \mathcal{P}$ if and only if $-\mathbf{x} \in \mathcal{P}$), it is straightforward that $\hat{1}_{\mathcal{P}}(\xi)$ is a real function of ξ. By an easy lemma, we know that the zero set of $\hat{1}_{\mathcal{P}}(\xi)$ contains a lattice of points if and only if $\mathcal{P}$ multiply tiles the plane with the set of translation vectors that comprise its dual lattice. We may therefore study the continuous curves that form the real zeros of $\hat{1}_{\mathcal{P}}(\xi)$, searching for lattices in them.

The graphs of these zero sets are quite beautiful, forming a countable, nested collection of continuous curves, and we begin to see some nice patterns in the plane; some of these patterns may vary from being easy to being to very challenging to prove. For example, is each of these compact continuous curves an algebraic curve? Some recent characterizations of multi-tiling polygons (and convex bodies in general) appear in the work of the second author, joint with N. Gravin and D. Shiryaev. But here we take a new direction.



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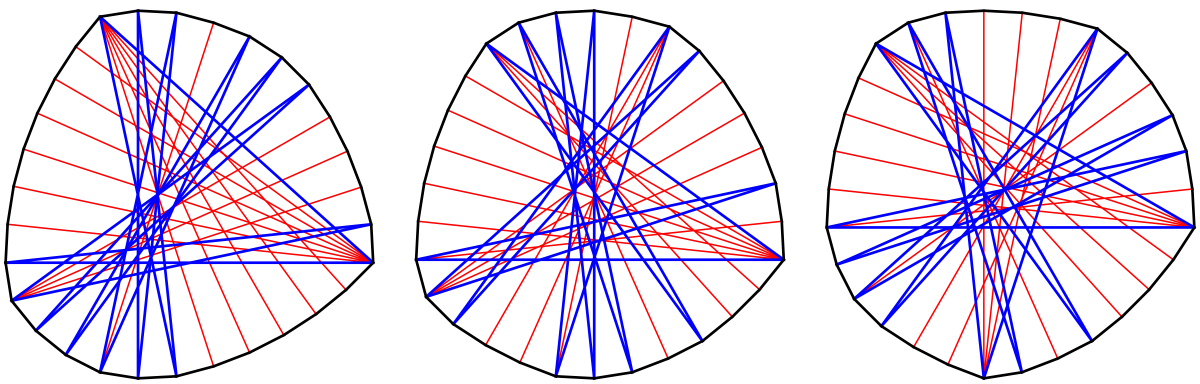
Reinhardt polygons. A *Reinhardt polygon* is a convex equilateral polygon in the plane with the property that if all of its vertices at maximal distance from one another in the polygon are connected with line segments, then a cycle appears among these line segments. For example, a regular $(2m+1)$-gon is a Reinhardt polygon, since the network of chords constructed in this case forms a regular star polygon with $2m+1$ points. Reinhardt polygons are extremal in a number of geometric problems, for instance, they have maximal perimeter relative to their diameter. Drawings of these polygons, with their internal chord structure shown, are quite striking. One may characterize Reinhardt polygons in terms of polynomials having a certain pattern in their coefficients, and having a zero at a particular root of unity. Some Reinhardt polygons, like the regular $(2m+1)$-gons, exhibit a particular pattern in their construction, and these are relatively well understood: if n is not a power of 2, then there are on the order of $2^{n/p}$ different Reinhardt n-gons exhibiting some kind of periodic pattern, where p is the smallest odd prime divisor of n. However, sporadic examples also occur: they first appear at $n = 30$, then at $n = 42$. Much is unknown about these, although it is known that the number of solutions depends on arithmetic properties of n. Some recent work of the first author, with K. Hare, shows that many sporadic examples exist for almost all integers n. These results were discovered after experimenting with certain kinds of combinations of cyclotomic polynomials. Much remains to explore, however, especially the case when n has three or more odd prime divisors.



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Computational aspects of Dedekind sums. A Dedekind sum is a very useful generalization of the greatest common divisor function. The 2-dimensional Dedekind sum, arising in the study of the enumeration of lattice points in rational polygons, may be defined by $s(a,b) := \sum_{k=1}^{b} B_1(k/b)B_1(ak/b)$, where $B_1(x) := x - \lfloor x \rfloor - \frac{1}{2}$ is the first periodic Bernoulli polynomial. Recently the second author studied the question of when two different Dedekind sums of the same fixed modulus m are equal, that is, for which a and c is it true that $s(a,m) = s(c,m)$? This question was motivated by a question in topology, coming from Lens spaces and studied by S. Jabuka. In recent joint work with S. Jabuka, X. Wang, and the second author, an answer was given when m is prime, but the problem is completely open for a general integer m. The answer turns out to have an application to correction terms in the recent development of the so-called Heegard-Floer homology. But here we will focus on the simple and computational aspects of the equality of the Dedekind sums, which is still open, as well as the equality of some of their natural generalizations.

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