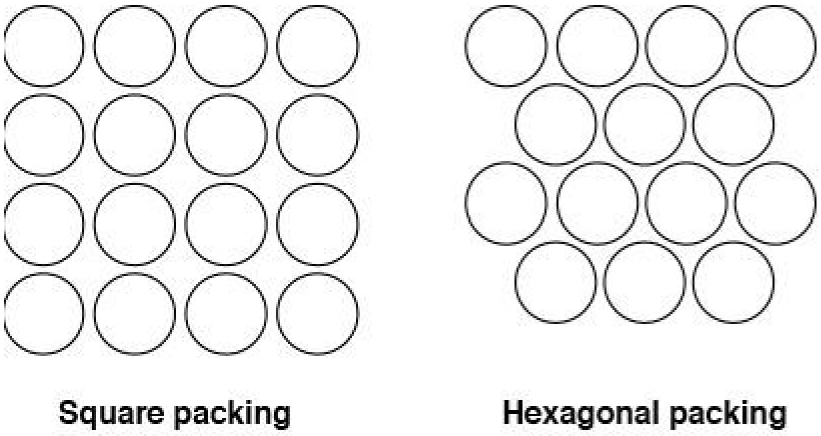
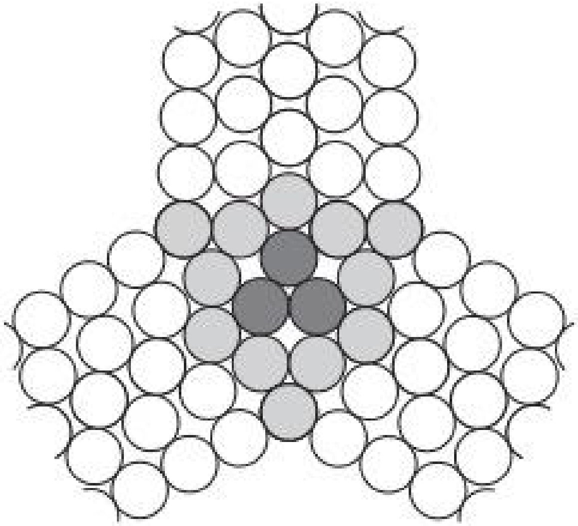
Configuration Spaces of Hard Disks

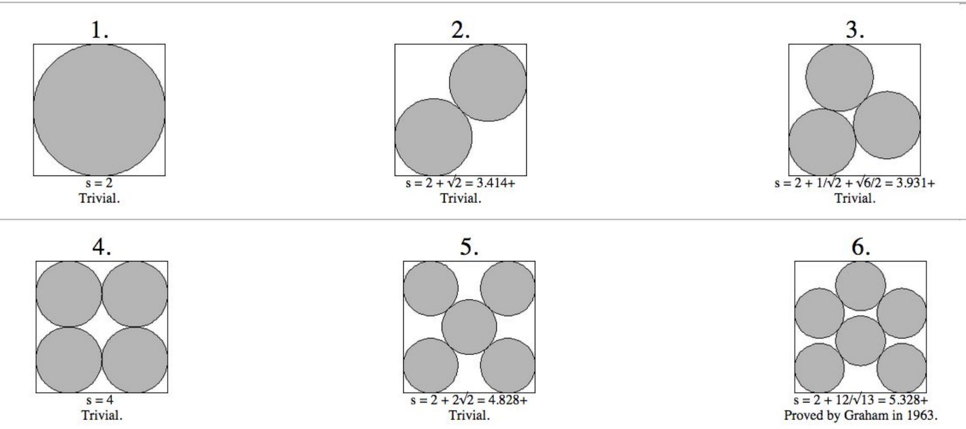
Emily Black and Esther Hunt

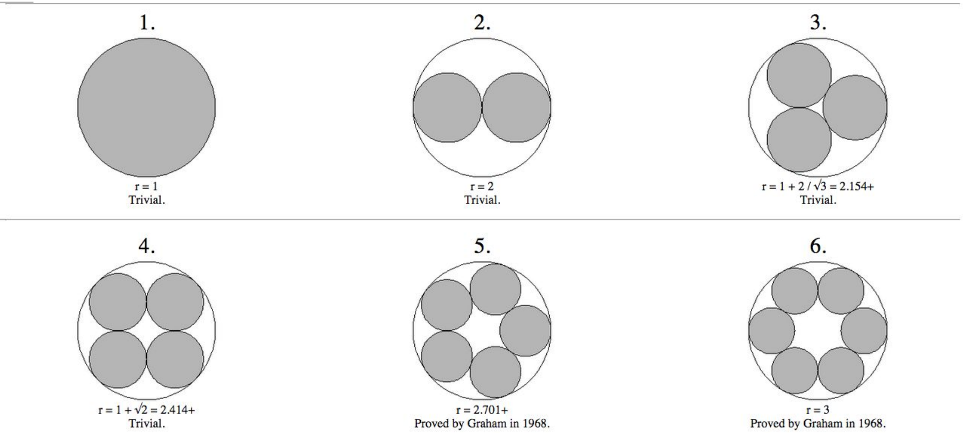
Packing Problems

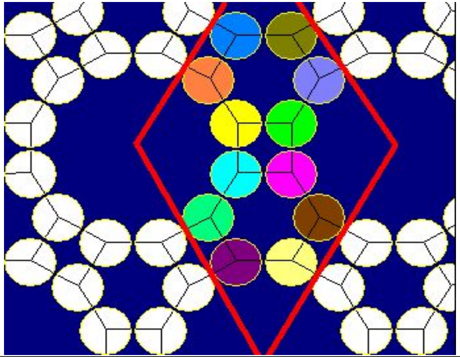


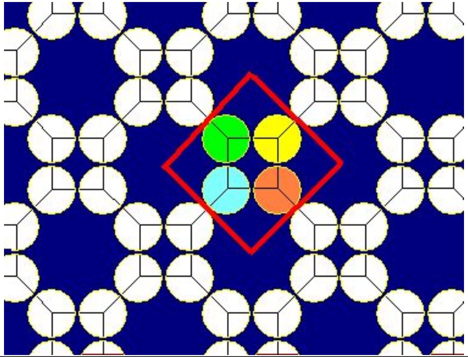


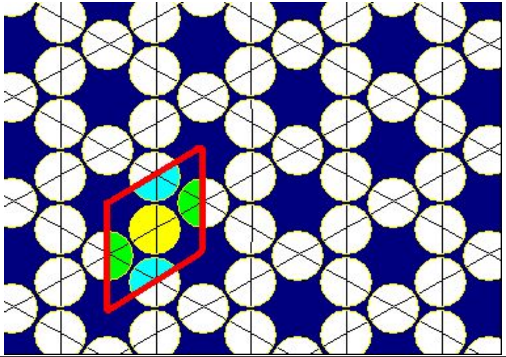
Images: http://matthewkahle.org/sites/default/files/papers/sparse%20stable.pdf
http://www.purplesquirrels.com.au/2015/02/css-hexagonal-packed-grid/









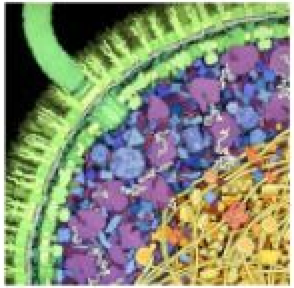


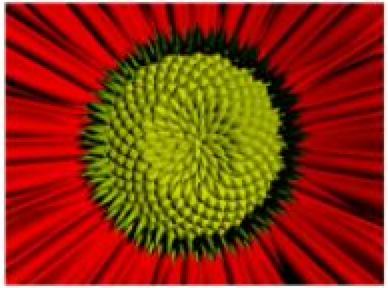
Sparse Versus Dense Packings

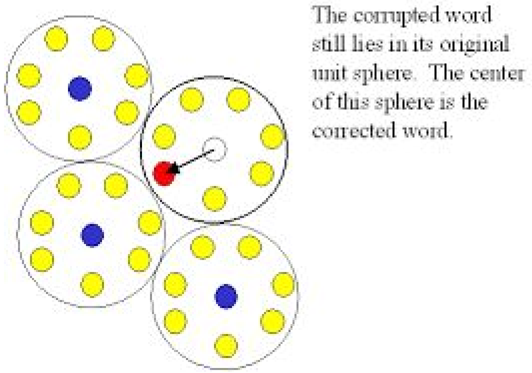
Images: http://www2.stetson.edu/~efriedma/packing.html
https://en.wikipedia.org/wiki/Circle_packing

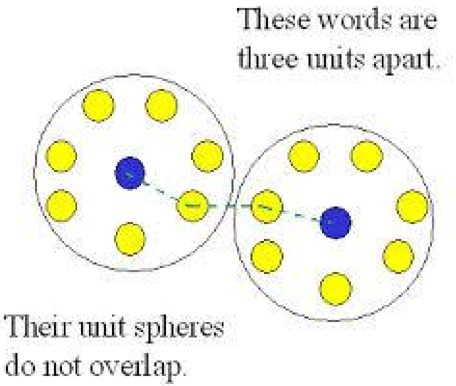
Applications of Dense Packings

- Properties of condensed-matter phases
- Coding theory
- “Crowding” of macromolecules within living cells
- Packing of cells to form tissue
- Competitive settlement of territories by animals









Images: http://www.quantdec.com/Articles/steganography/ecc.htm
https://www.princeton.edu/~fhs/paper349/paper349.pdf

Applications of Sparse Packings

Boltzmann Gases

- Particles move freely with little interaction
- Is every configuration attainable from any other configuration?
- Is the Configuration Space connected?

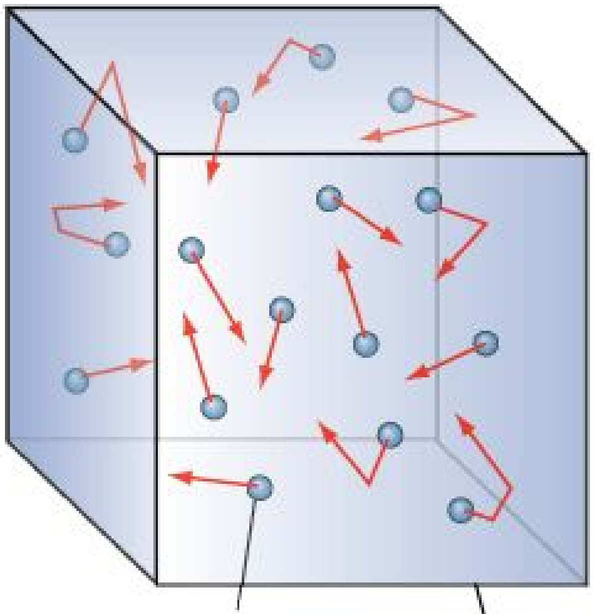
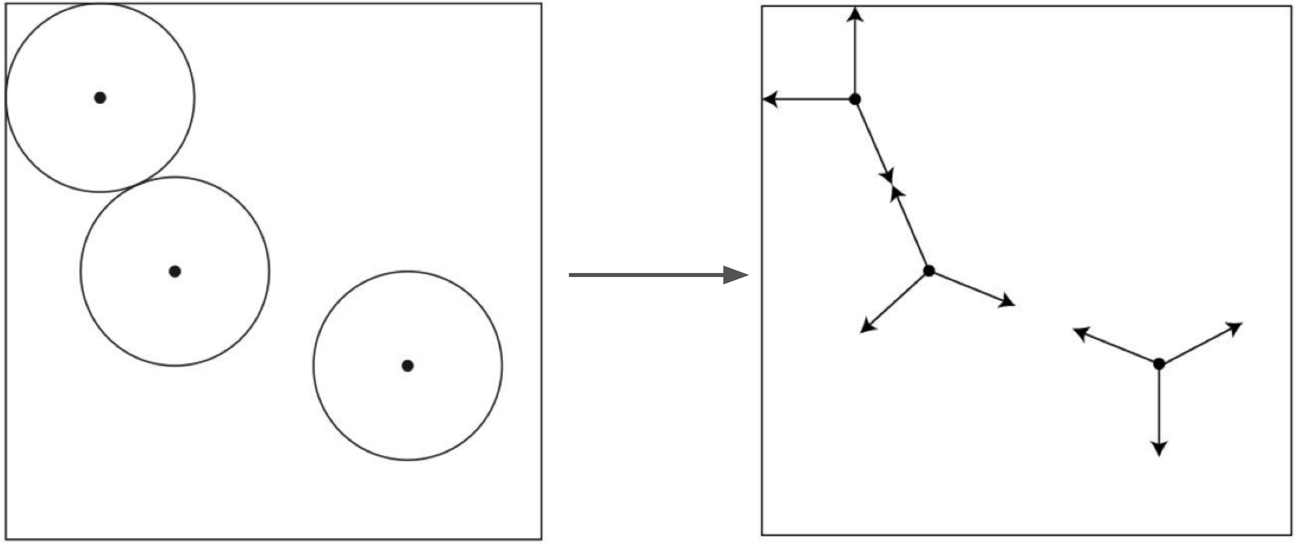


Image: http://www.daviddarling.info/encyclopedia/G/gas.html

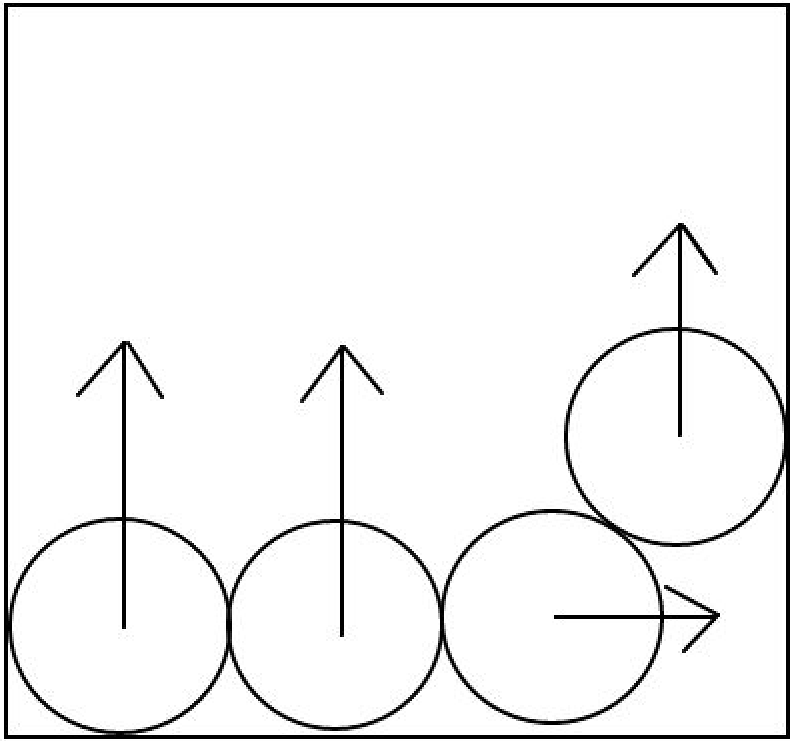
What Are Configuration Spaces of Hard Disks?

The configuration space of n hard disks is a subset of R^{2n}.
Tau: R^{2n}-->R; Configuration-->maximum possible radius



What Does it Mean to Lock?

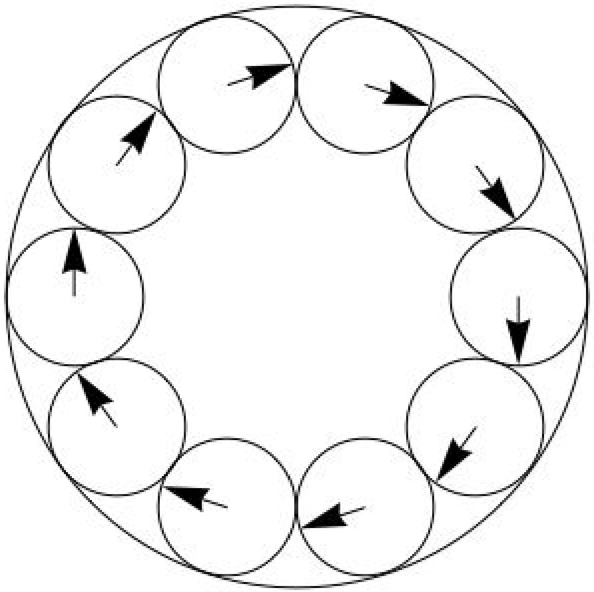
Unlocked: The balls can move freely to change configuration



What Does it Mean to Lock?

Locally Locked/Jammed:

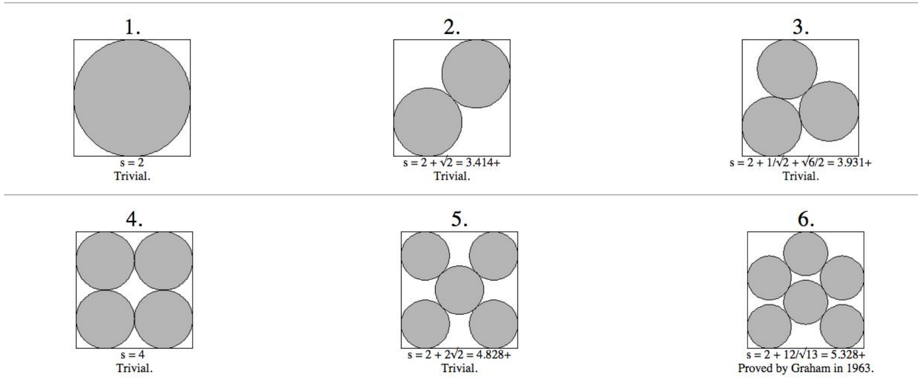
- each disk in the packing is locally trapped by its neighbors
- ie it has at least 3 contacts
- cannot be moved while fixing the positions of all other particles



What Does it Mean to Lock?

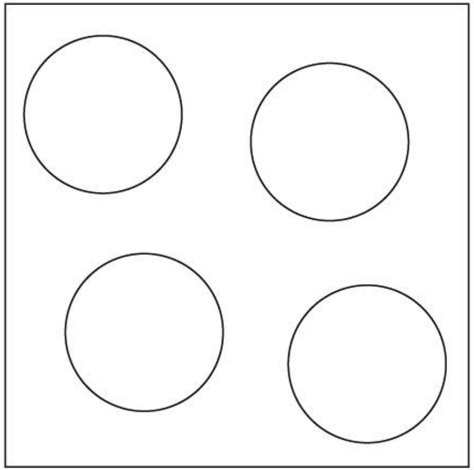
Globally Locked:

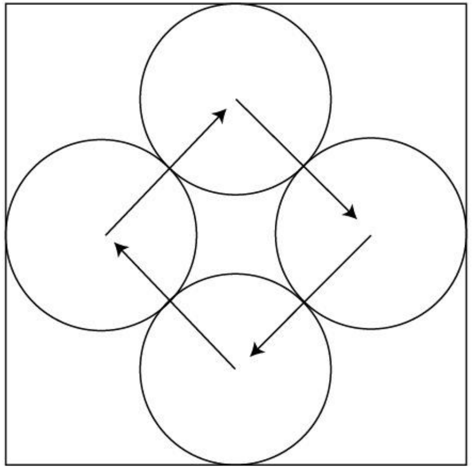
- No subset of the balls can move at all. One example would be a lattice packing.
- Global lockings refer to a local maximum of Tau.

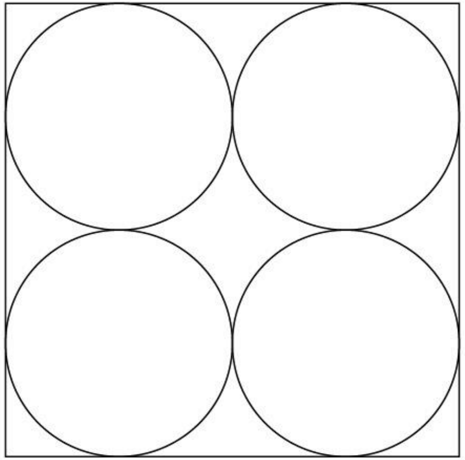


http://www2.stetson.edu/~efriedma/packing.html

What Does it Mean to Lock?

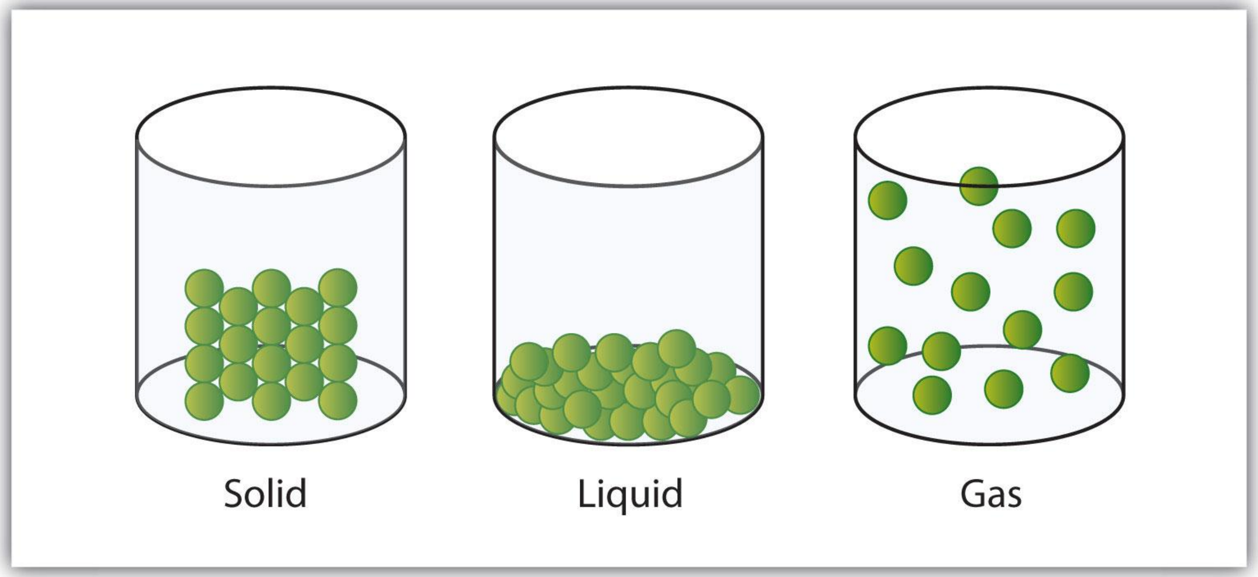

Unlocked

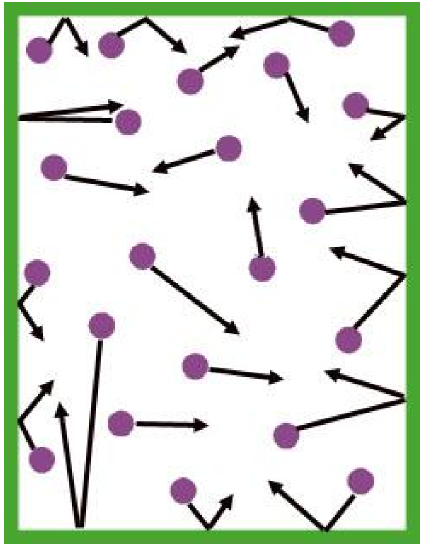

Locally Locked


Globally Locked

Sparse Packings: Things To Note

$$n \longrightarrow \infty\ ;\quad nr^2 \longrightarrow 0$$

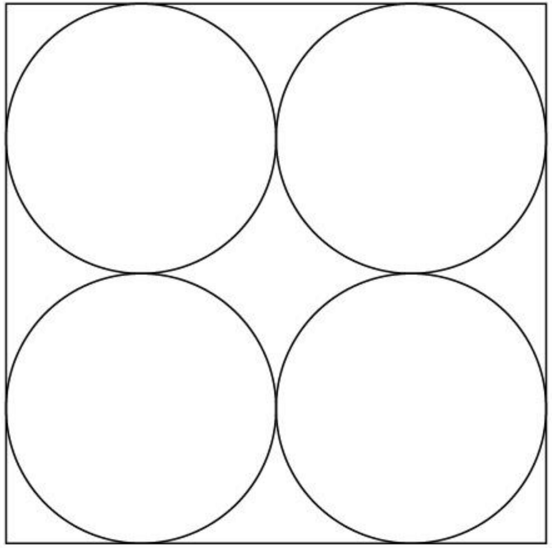




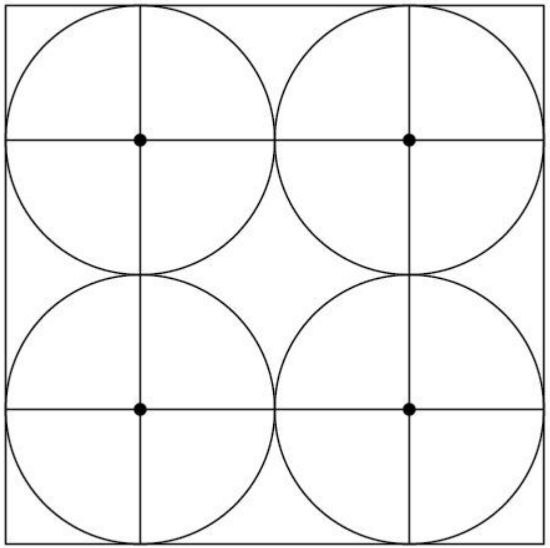
Images: http://www.chemistry.wustl.edu/~edudev/LabTutorials/Airbags/airbags.html
http://www.oxnotes.com/states-of-matter-igcse-chemistry.html

Configurations: Stress Graphs

Nodes: disk centers

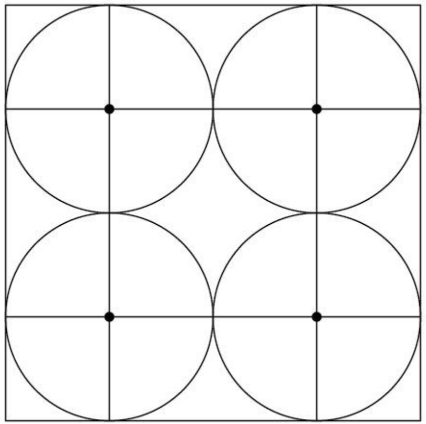


Edges: disk contacts

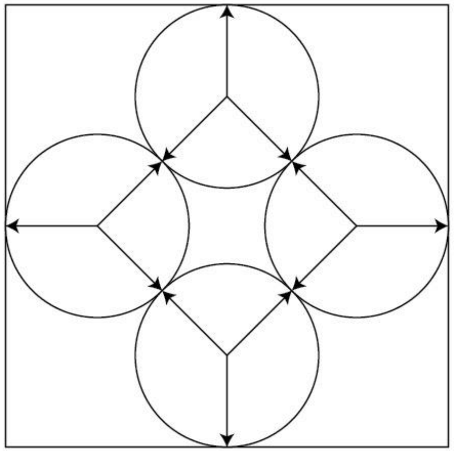


Connelly Theorem

Theorem: if there exists an infinitesimal, local motion given by a 2n-component vector v in a concave or polygonal shape, then there is a global unlocking motion of the configuration.



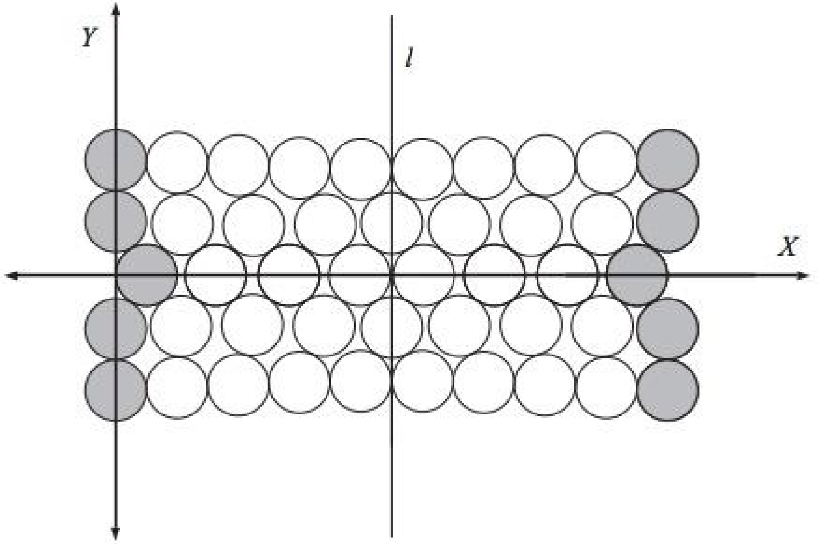
4 Disks, 12 Contacts

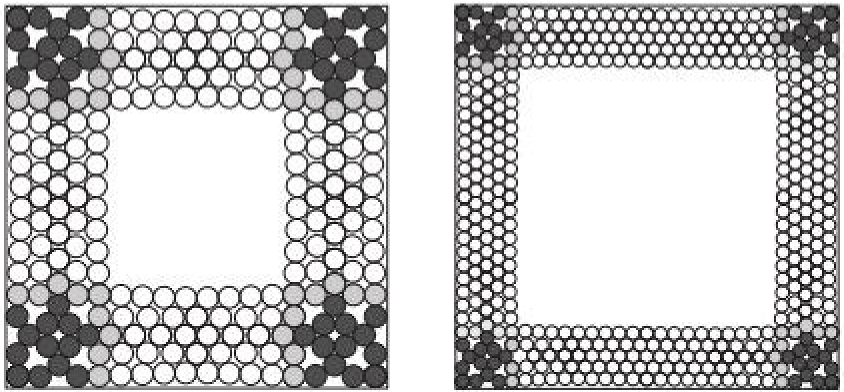


4 disks, 8 contacts

Surprising Unlocked Configurations

The Boroczky Bridge

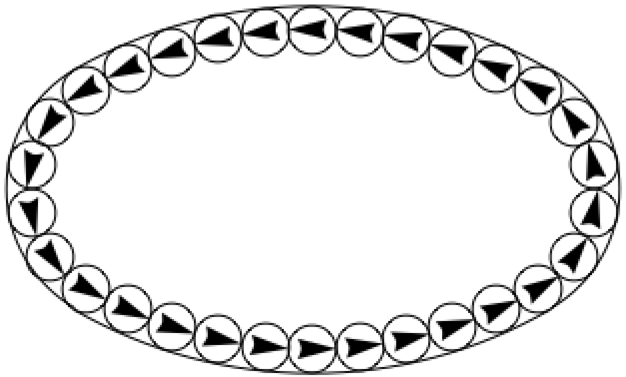




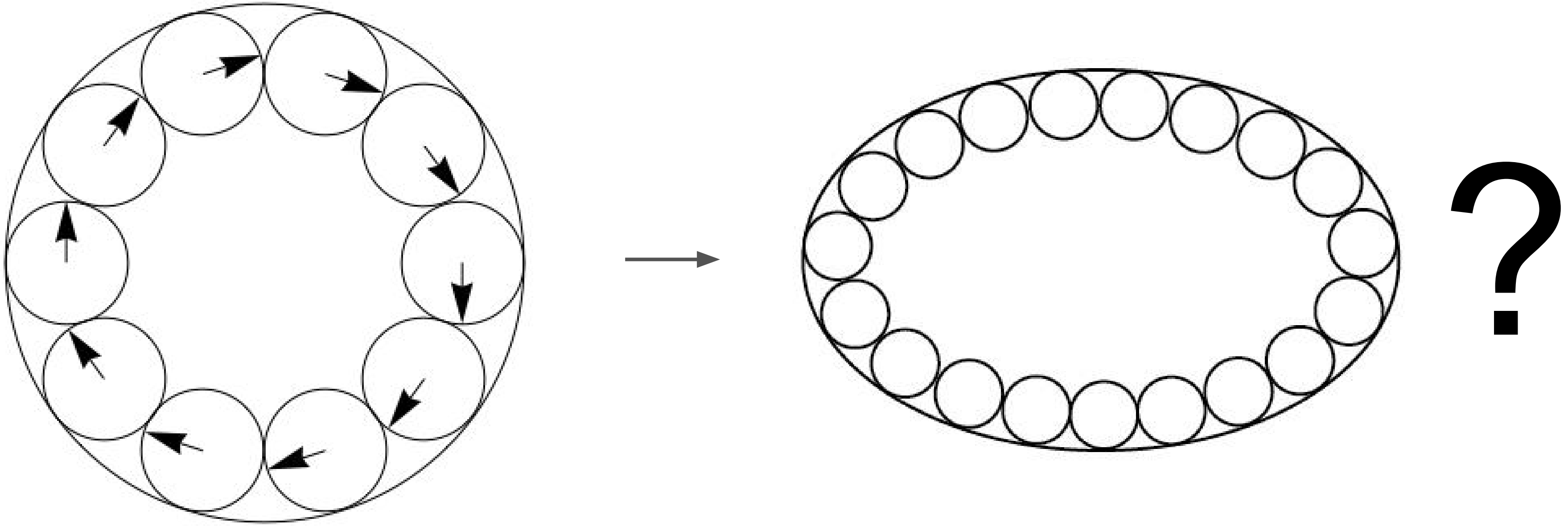
Images: http://matthewkahle.org/sites/default/files/papers/sparse%20stable.pdf

Code Based on Linear Inequalities

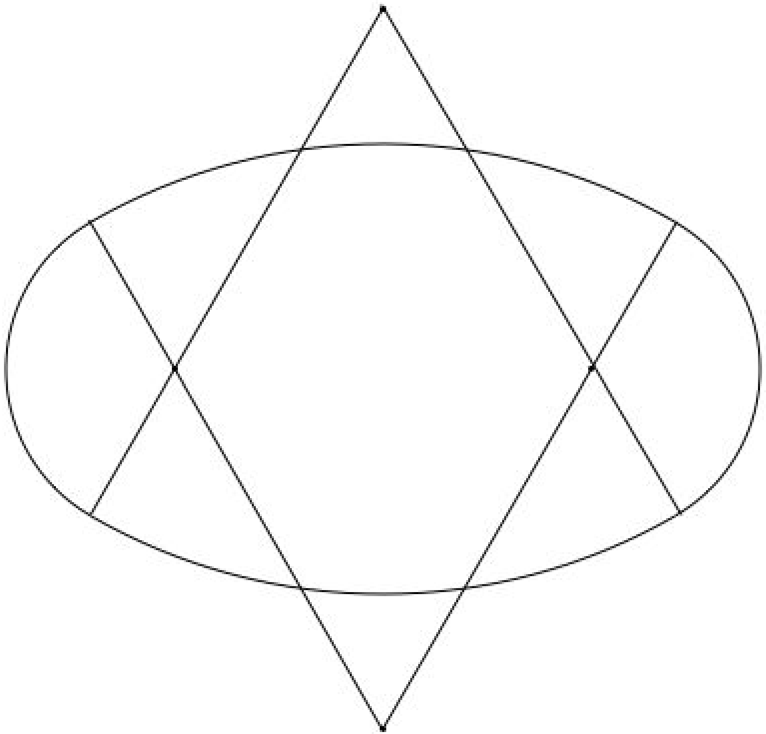
Pull up Mathematica

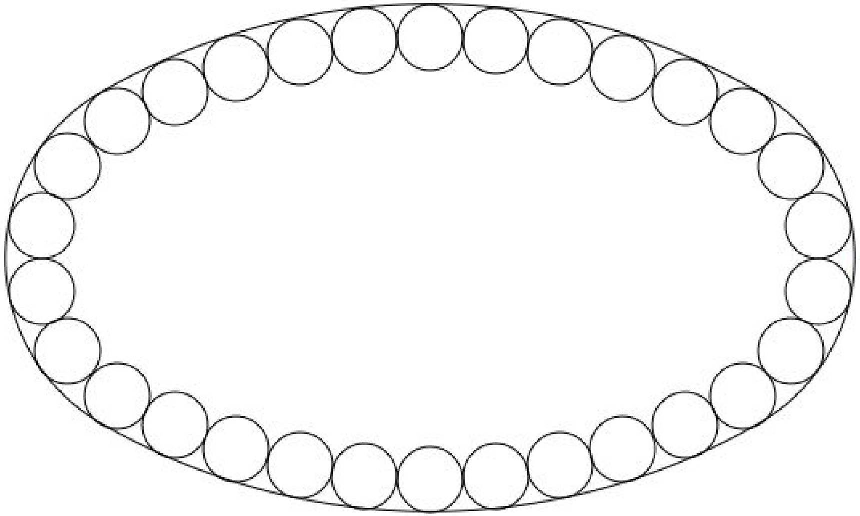


Convex Shapes: The Intuition



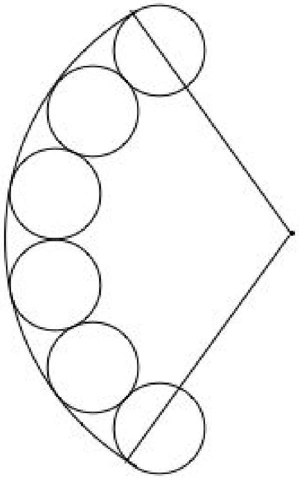
The Intuition Breaks: Our Shape

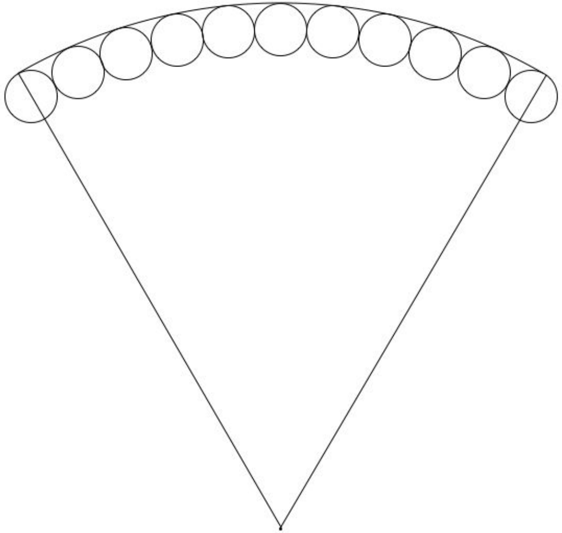




Construction: R, r, alpha, m, and n

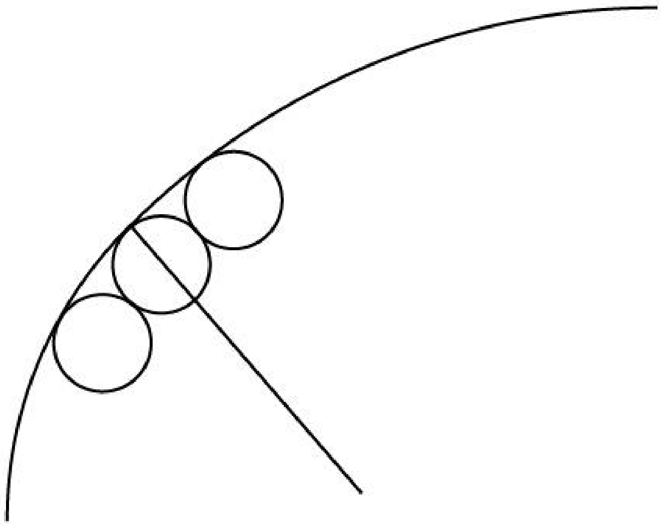
We only want R and r such that the arcs contain an integer number of m and n balls

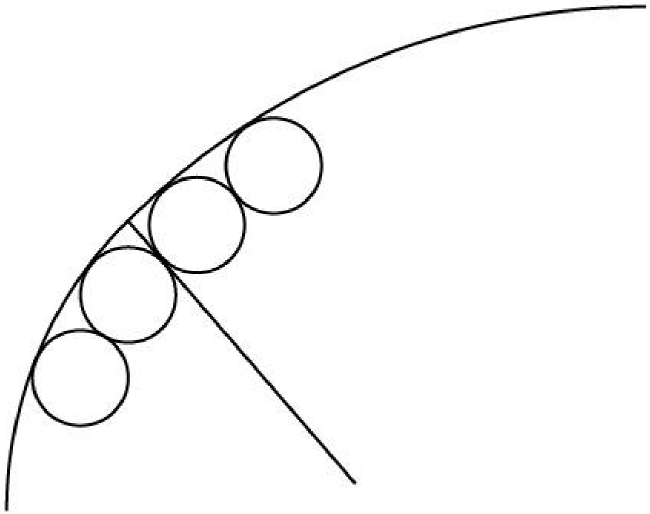




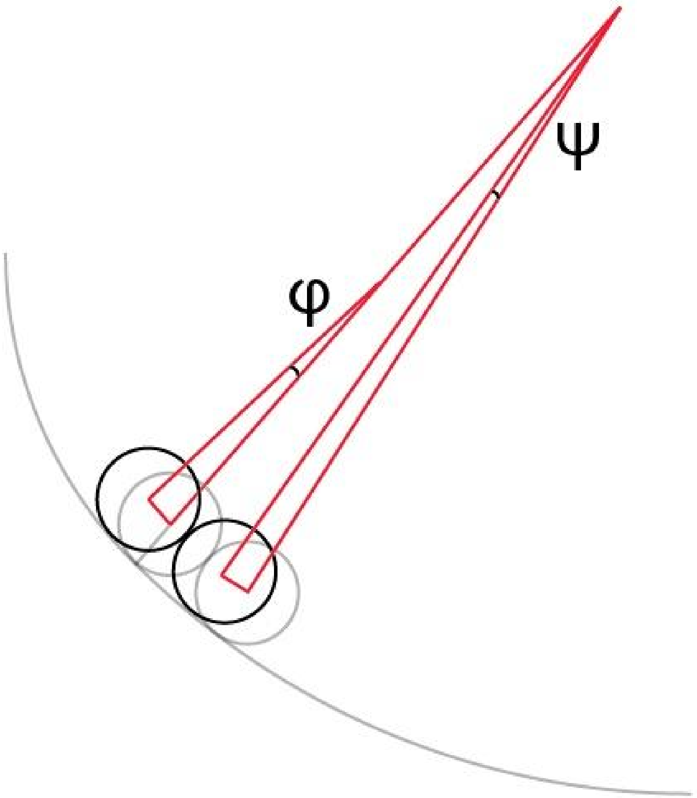
Unlocking and Locking Possibilities

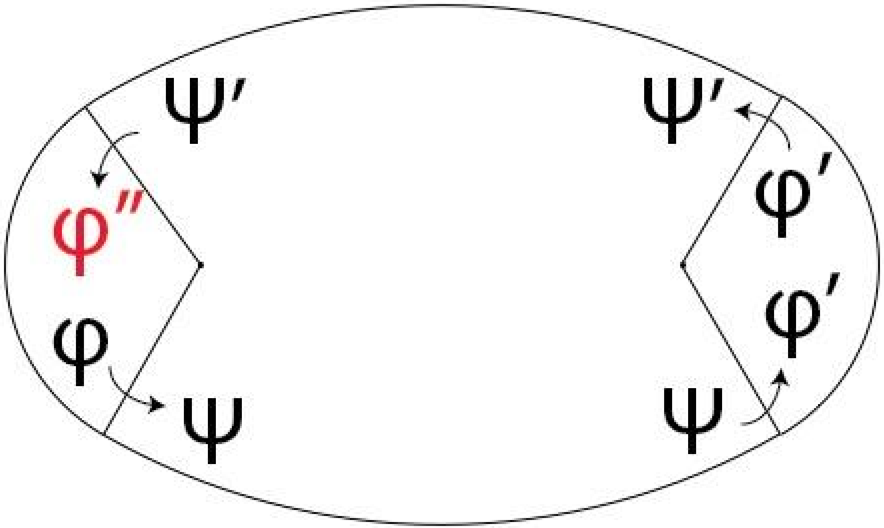
Two Different Starting Points and Intuitions





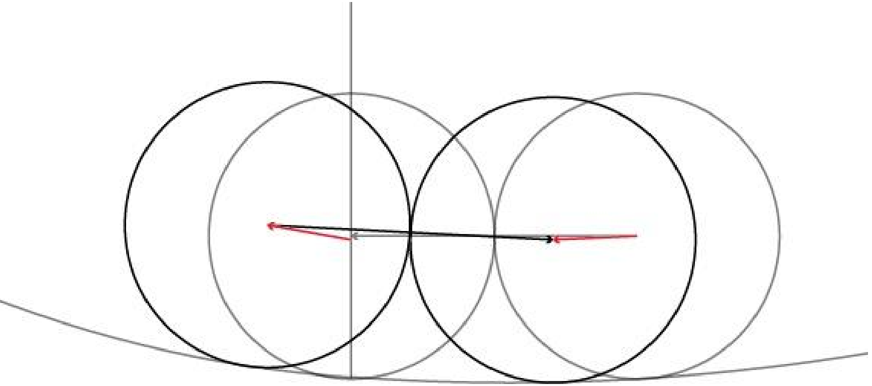
Displacement Around The Shape

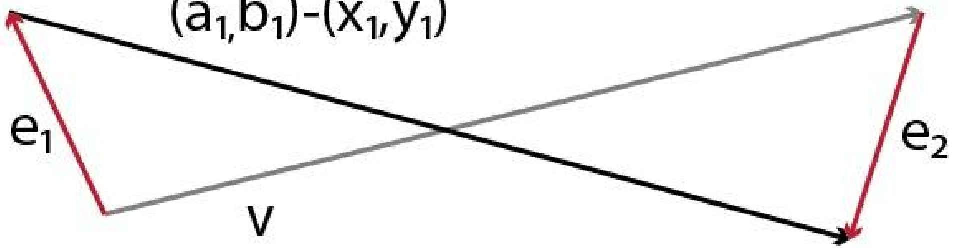




Getting an Equation

$$\varphi'' = c_0 + c_1\varphi + c_2\varphi^2 + \ldots$$

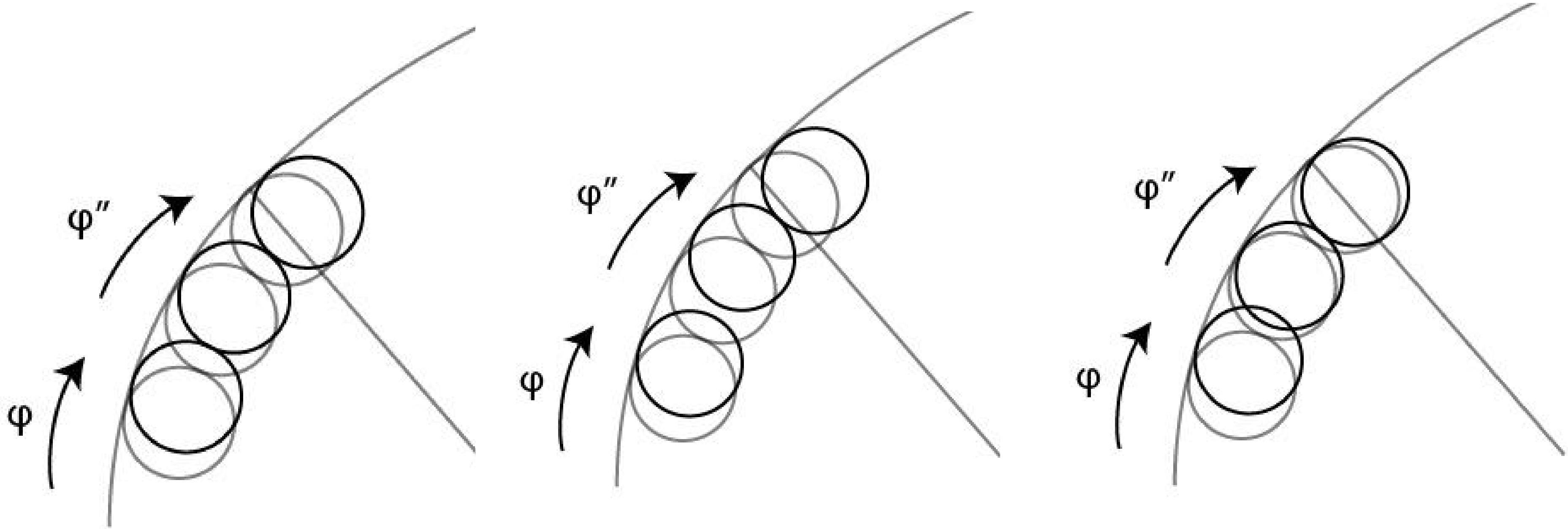




$$|(a_1,b_1) - (x_1,y_1)|^2 = 4$$

$$|e_1 - V - e_2|^2 = 4$$

Sign of the Quadratic Term



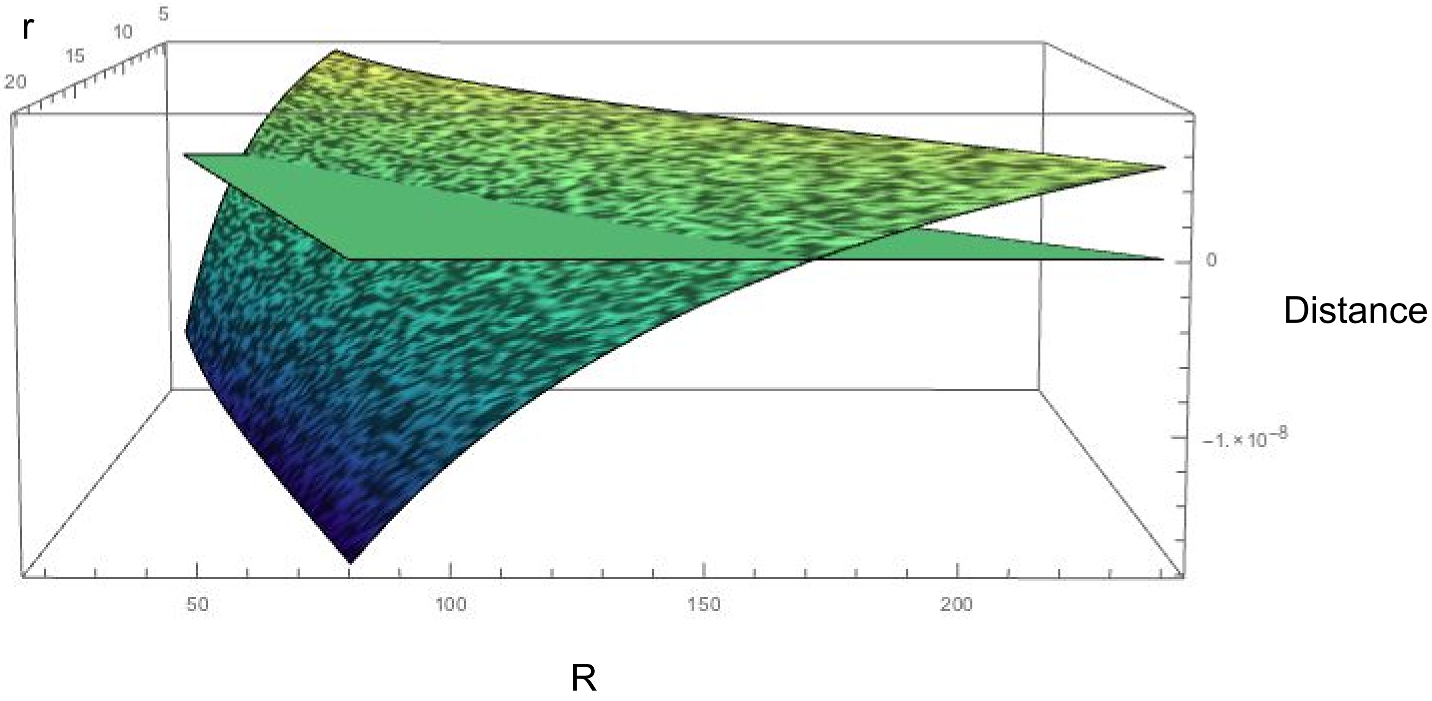
Our Results

Theorem BH

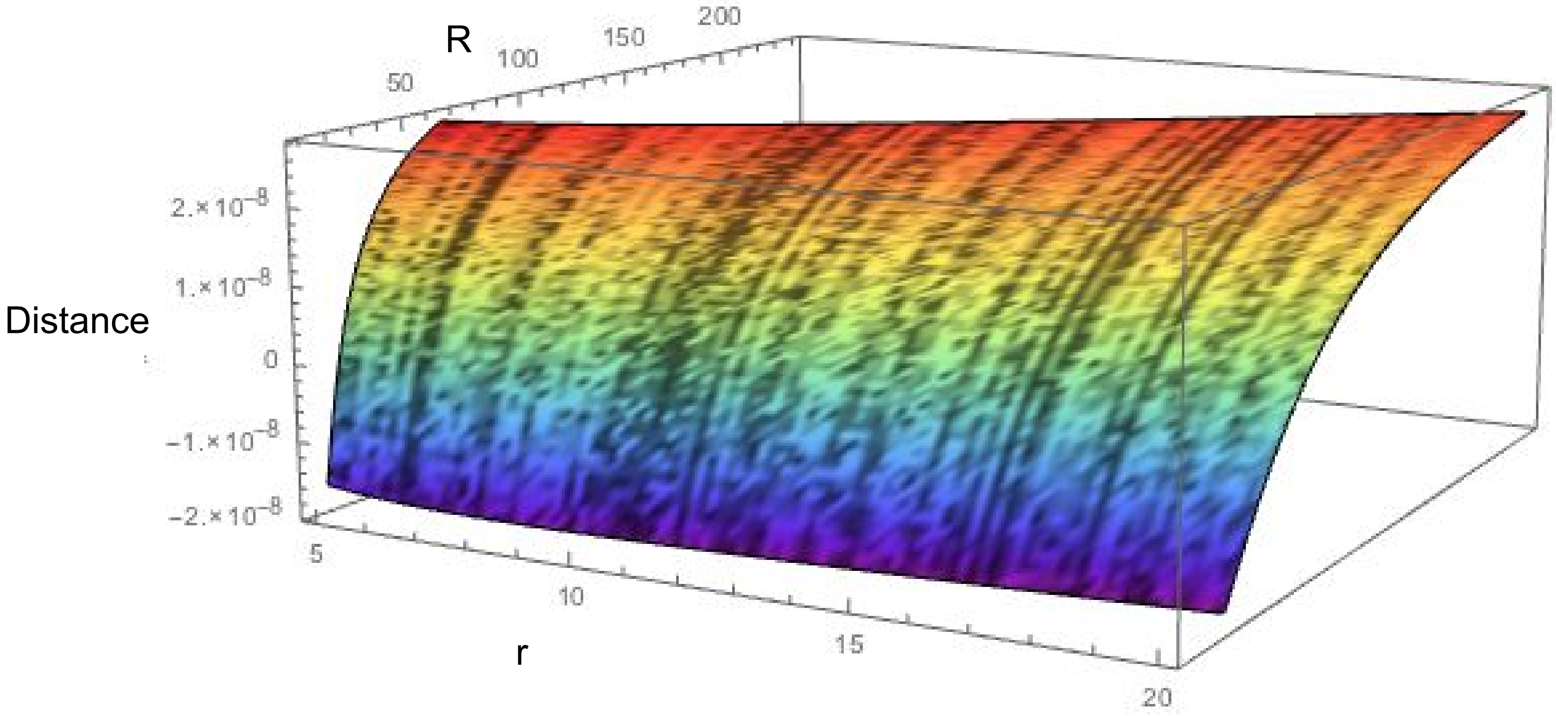
The smooth two-arc configuration is determined by three parameters, r, R, and alpha. However, whether or not the configuration locks for integer number of balls n and m, on the small and large arcs, depends only upon r and R, or, equivalently, n and m.

$$\varphi'' = p - \frac{p^2 (r-R)\left((R-1)^2 - 4\sqrt{(r-2)\,r}\,\sqrt{(R-2)\,R}\right)}{\sqrt{(r-2)\,r}\,(R-1)^3}$$

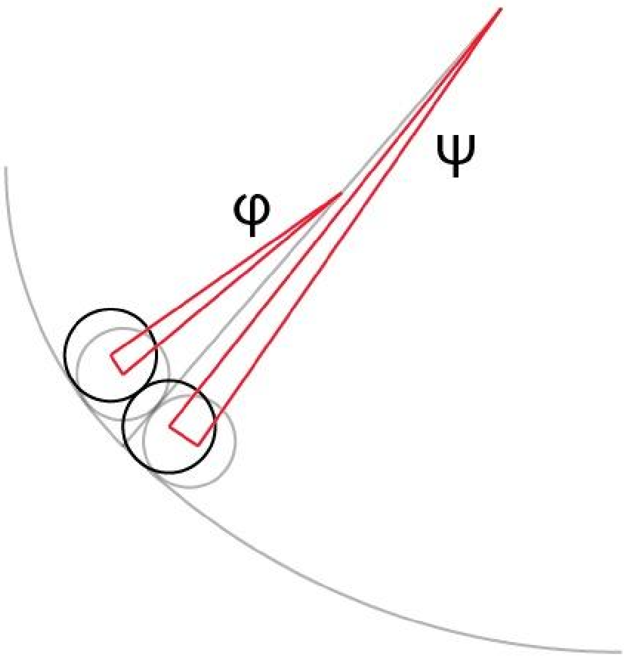
Our Results



Our Results



Non-Overlapping Starting Position

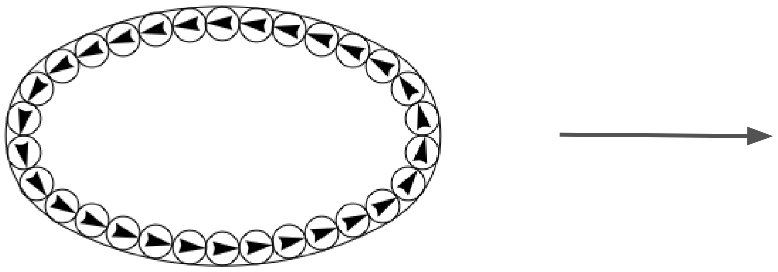


Work So Far: Linear Term

$$\psi = \varphi \frac{(r-1)((R-1)\sin(\theta+\gamma) - (R-r)\sin\gamma)}{(R-1)((r-1)\sin(\theta+\gamma) + (R-r)\sin\theta)}$$

Investigating Unlocking Situations

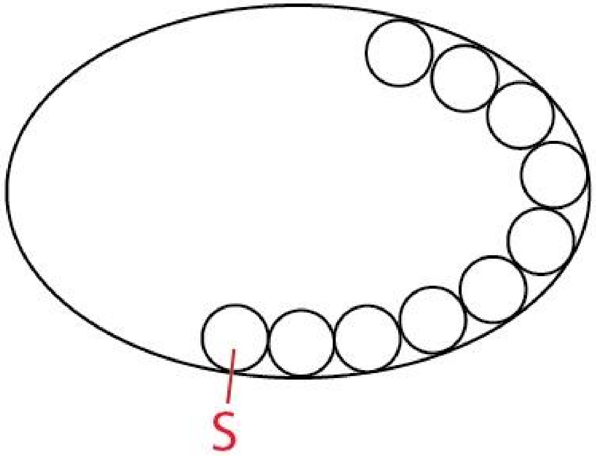
Code needs to be so that we get a global unlocking motion instead of an infinitesimal, local motion.

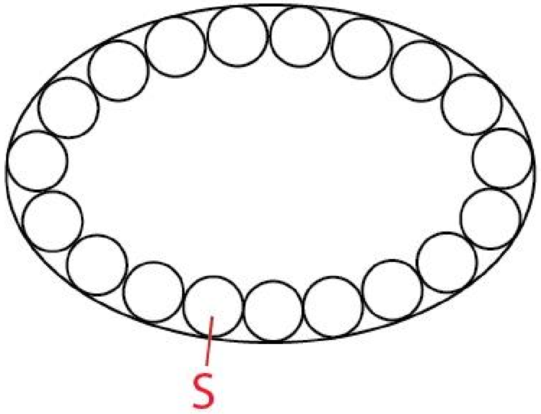


Motion goes along entire shape, and hopefully one ball pops out

Further Directions: Proving Locking

Lemma: If $rho_n(s)$, where s is a starting point on the shape in question, is not a constant function, then there exists a locking configuration on the shape





Acknowledgements

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Questions/Comments?

Thank you for listening!