CYCLIC EVASION IN THE FOUR BUG PROBLEM

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Objective

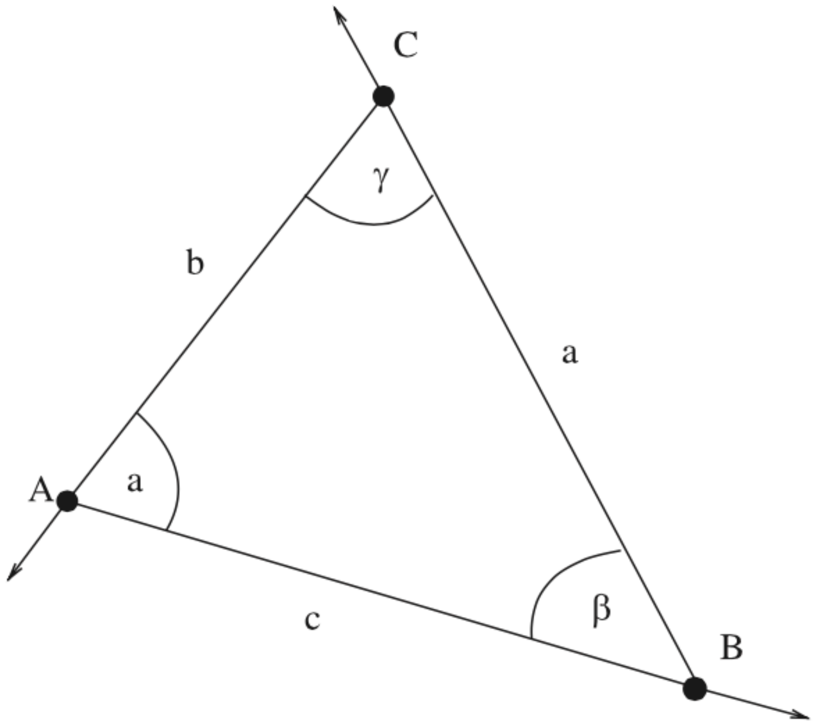
To determine the long term behavior of any four bug configuration

Outline

- I. Background
- II. Four Bug Problem
- III. Fixed Point Analysis
- IV. Stability Analysis
- V. Conclusion

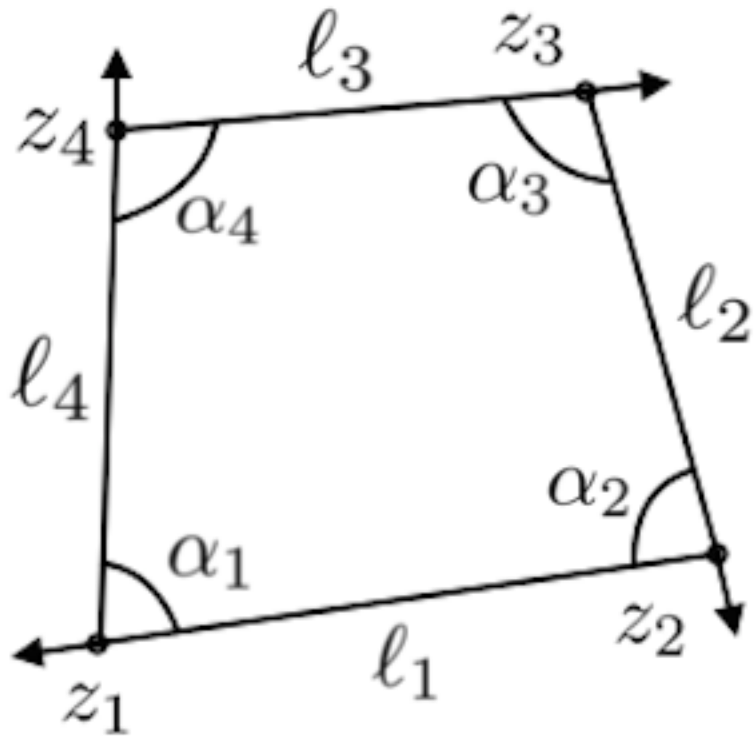
Background

- Pursuit and Evasion
- Three Bug Problem
  - Equilateral Triangles
- N-Bug Problem
  - Stable configurations



Four Bug Problem

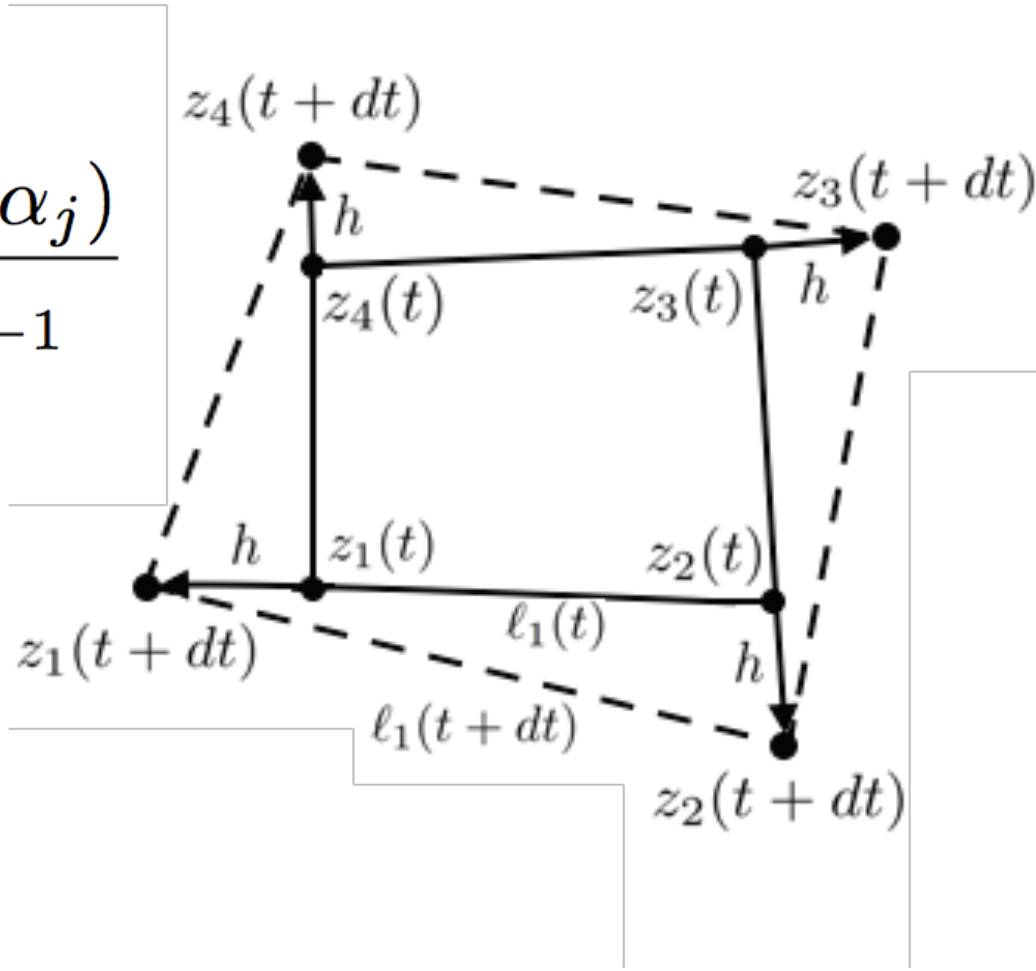
$$\frac{d\mathbf{z_j}}{dt} = \frac{\mathbf{z_j} - \mathbf{z_{j+1}}}{|\mathbf{z_j} - \mathbf{z_{j+1}}|}$$



Derivation

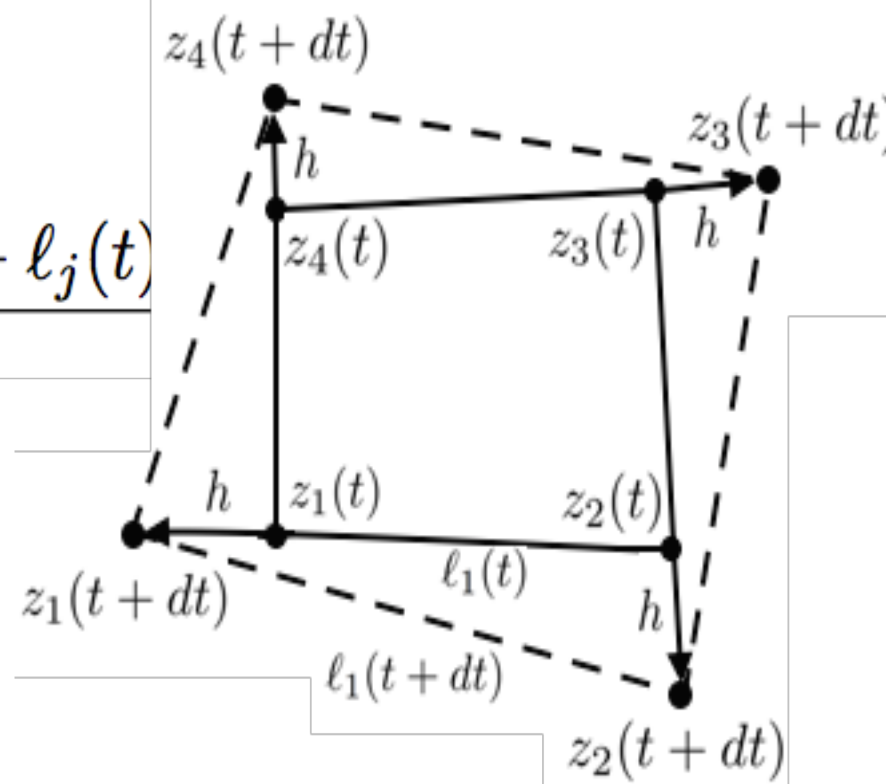
$$\dot{\alpha}_j = \frac{\sin(\alpha_{j+1})}{\ell_j} - \frac{\sin(\alpha_j)}{\ell_{j-1}}$$

$$\dot{\ell}_j = 1 + \cos(\alpha_{j+1})$$



Derivation

$$
\begin{aligned}
\dot{\ell}_j &= \lim_{h\to 0} \frac{\ell_j(t+h) - \ell_j(t)}{h} \\
&= \lim_{h\to 0} \frac{\ell_j(t) + h + h\cos(\alpha_{j+1}) - \ell_j(t)}{h} \\
&= \lim_{h\to 0} \frac{h + h\cos(\alpha_{j+1})}{h} \\
&= 1 + \cos(\alpha_{j+1})
\end{aligned}
$$



Fixed Points

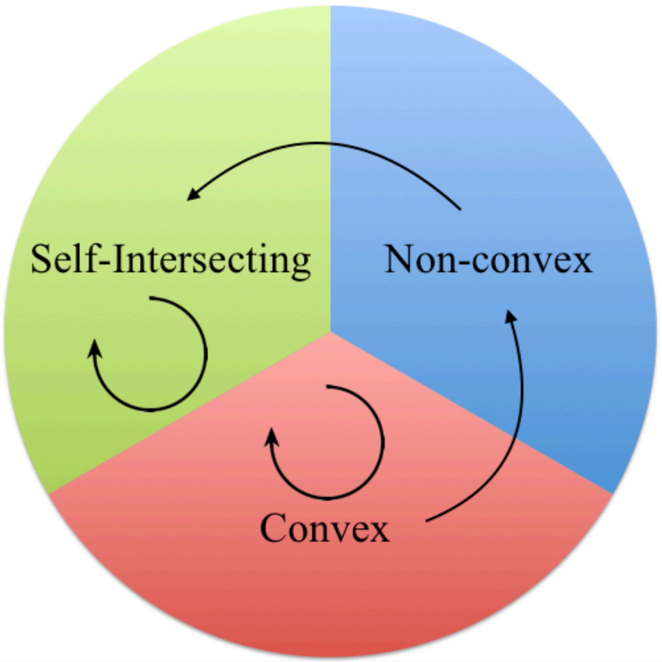
Theorem 1.1: *The only fixed configurations for any four bug configuration are the self-intersecting line and square configuration*

Shape Evolution

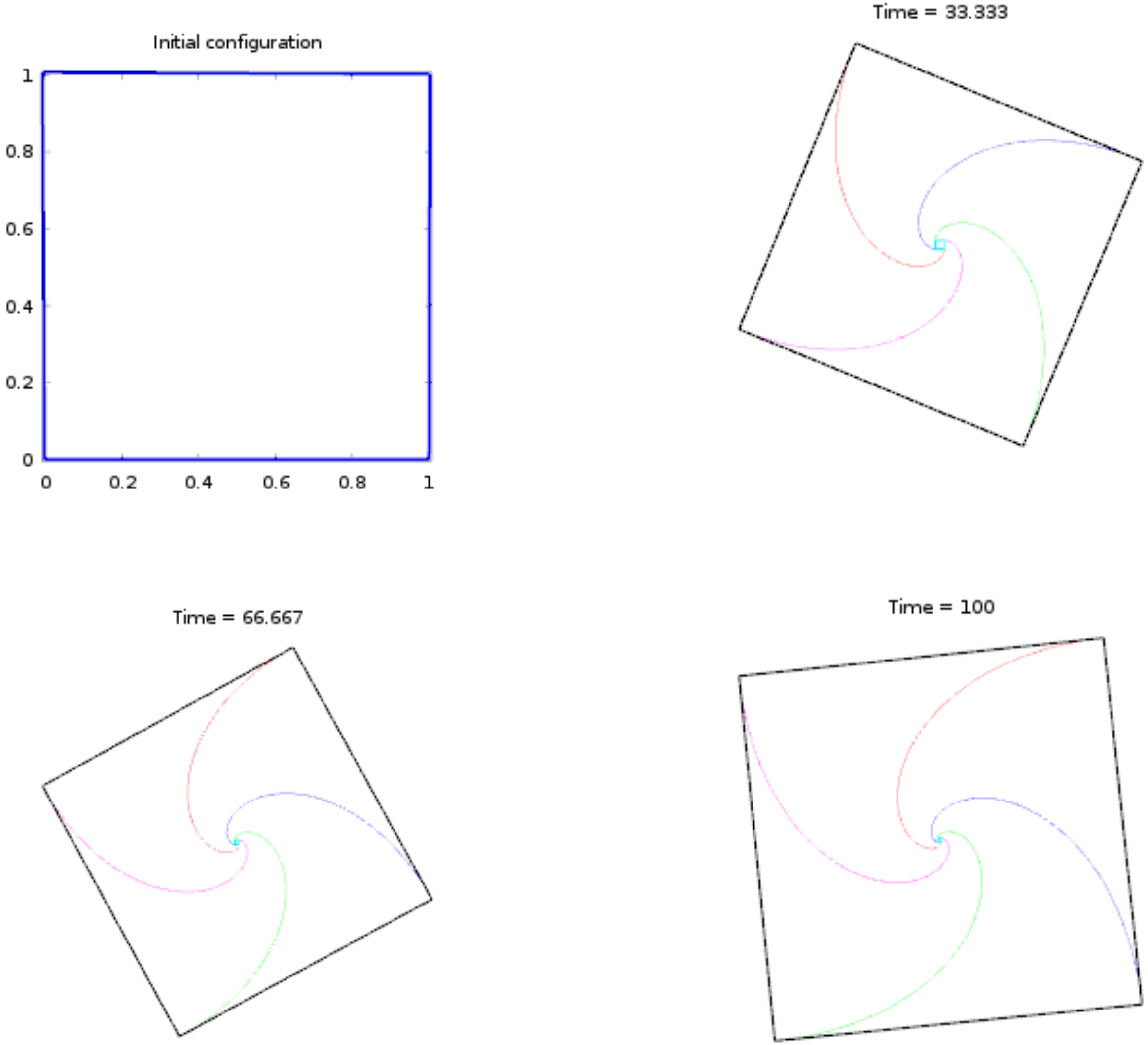
I. Convex
   - Convex
   - Self-intersecting

II. Non-convex
   - Self-intersecting

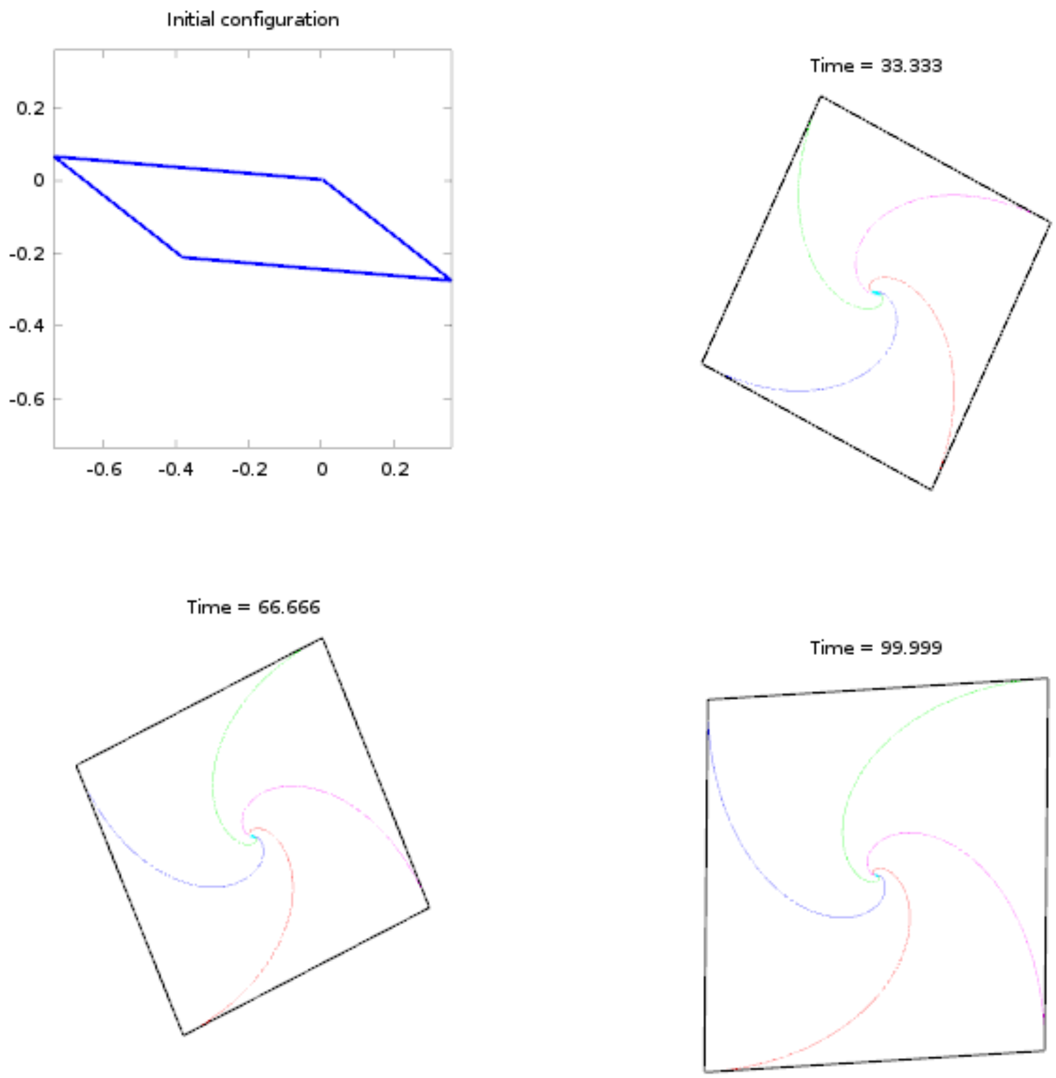
III. Self-Intersecting
   - Self-intersecting



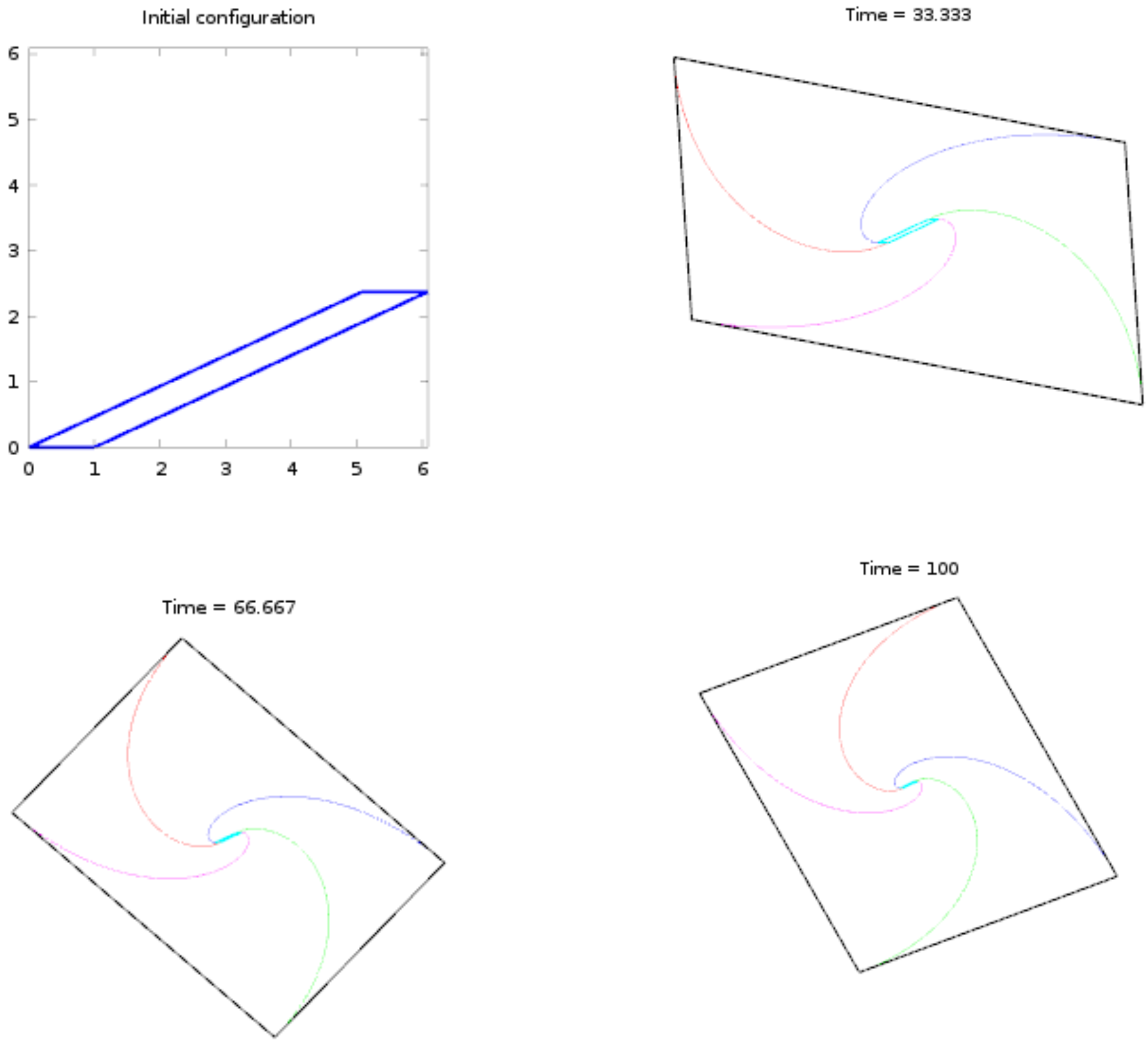
Evolution of a Square



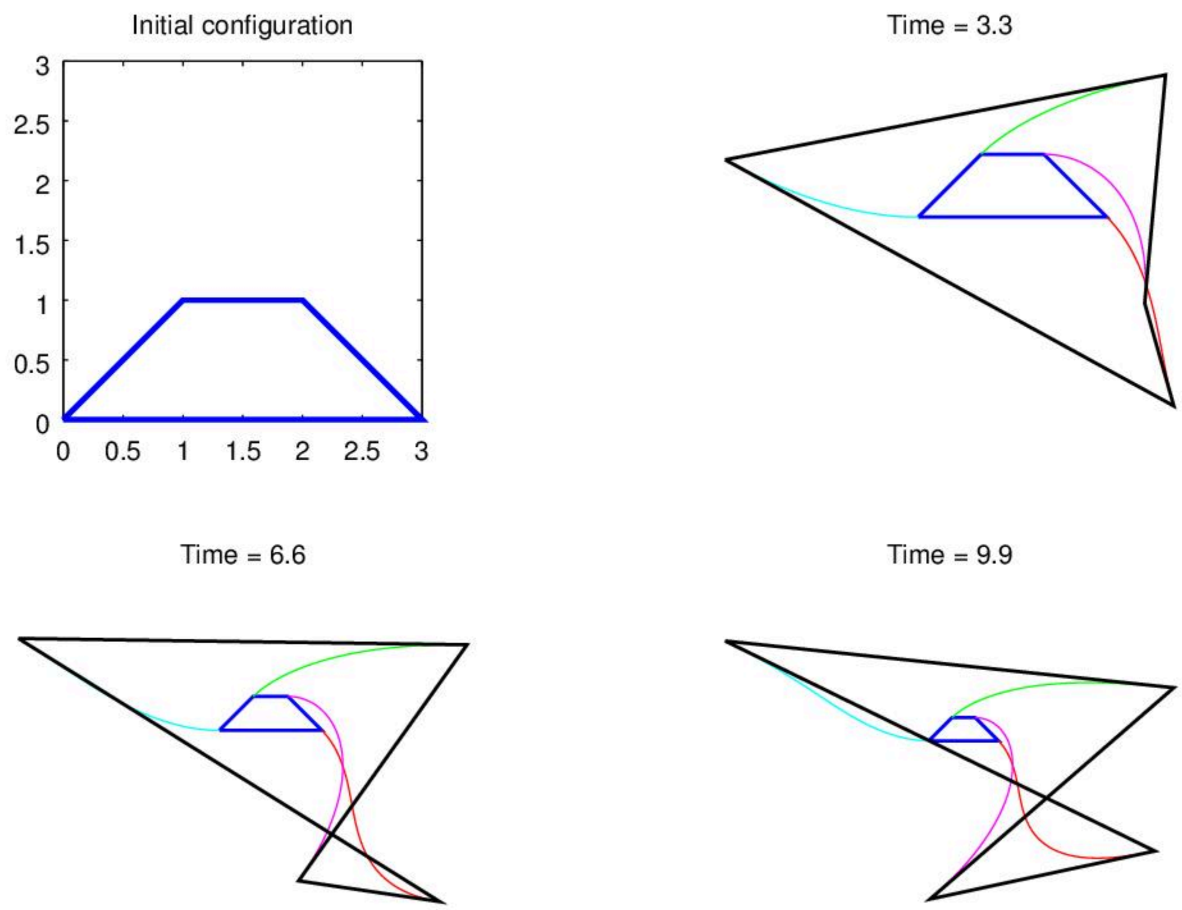
Evolution of a Parallelogram



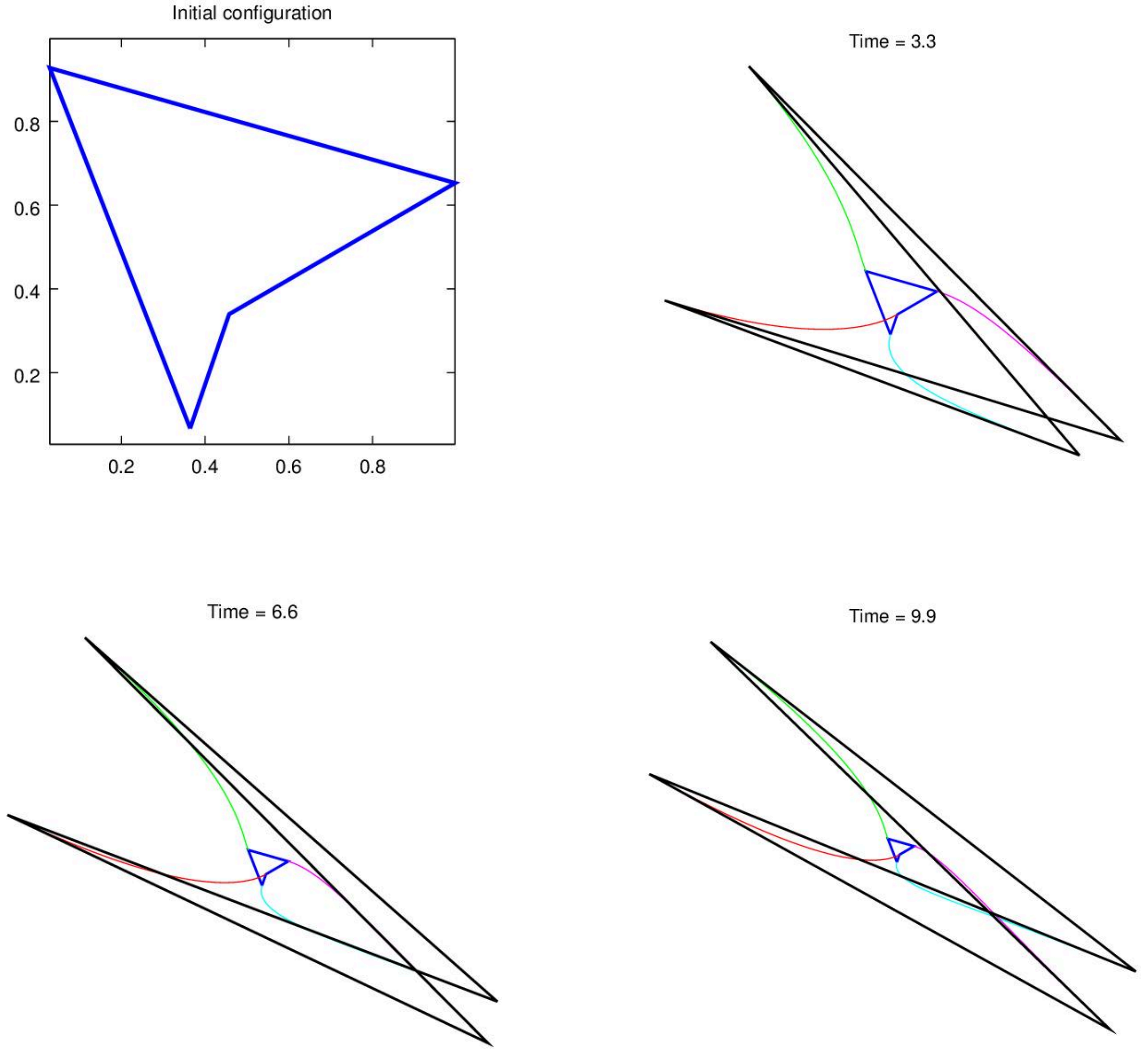
Evolution of a Parallelogram



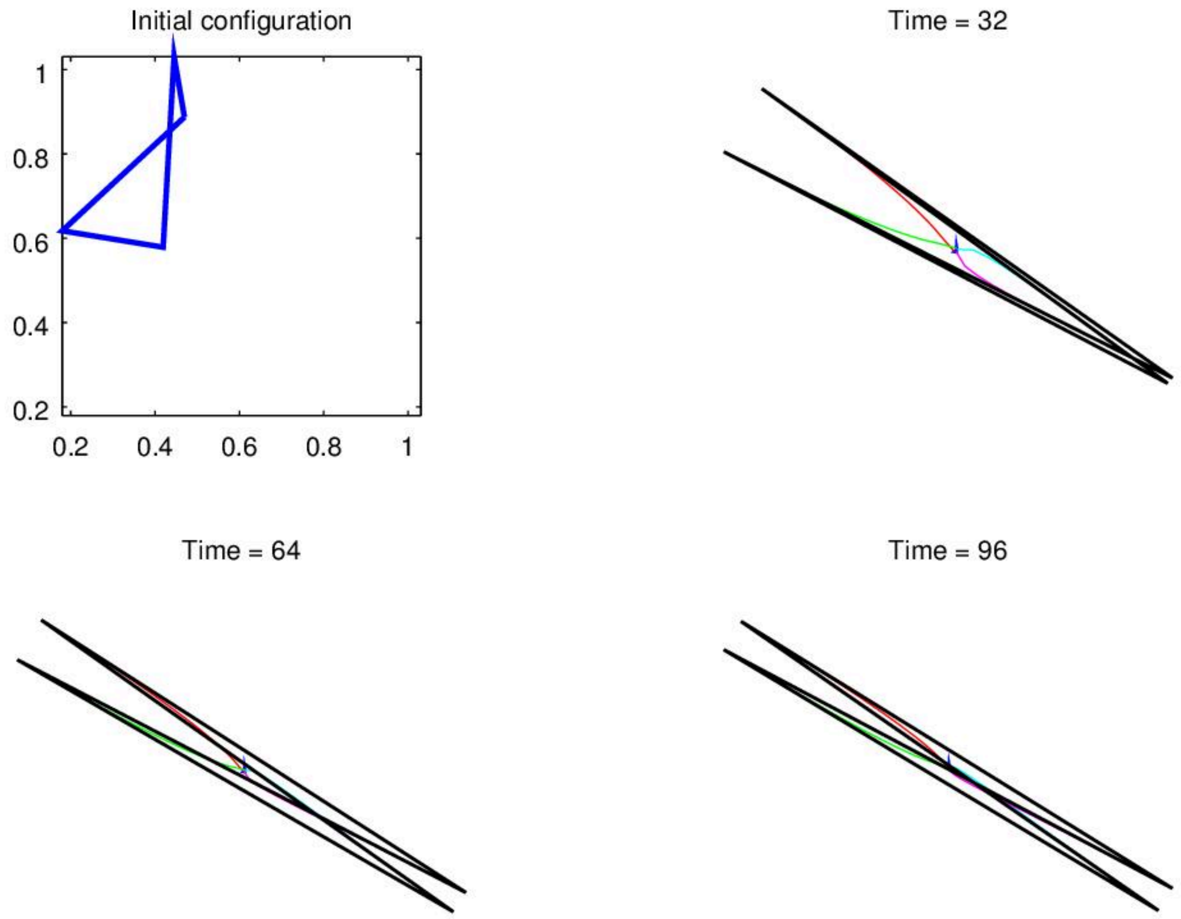
Evolution of a Convex Configuration



Evolution of a Non-convex Configuration



Evolution of Self-Intersecting Configuration



Fixed Point Analysis

I. Linearization

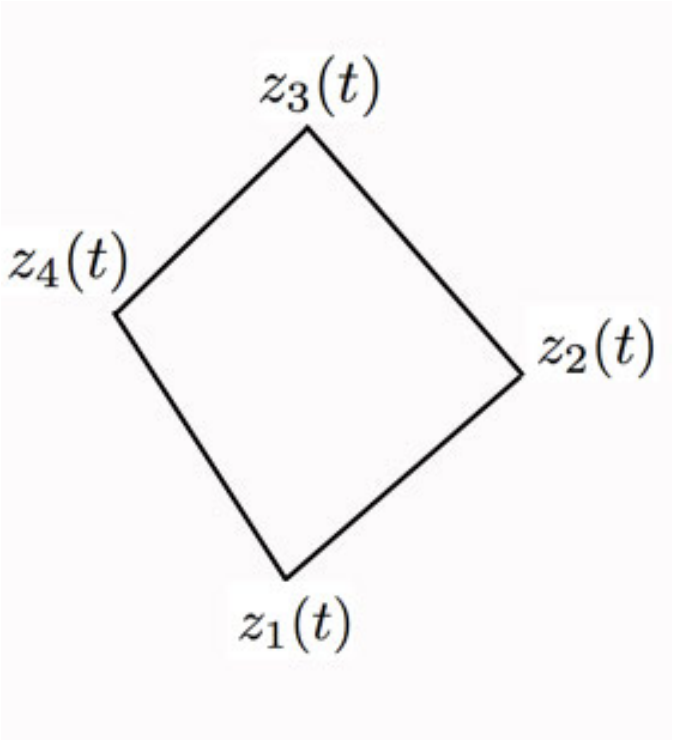
Let $z_j(t) \in \mathbb{C}$, such that $t \geq 0$ and

$$z_1(t) = 0 + 0i$$

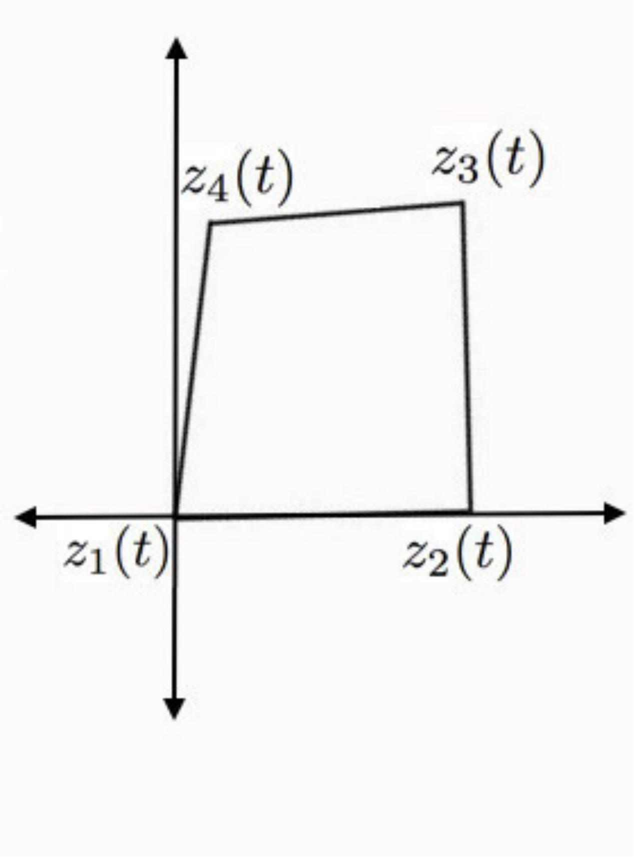
$$z_2(t) = 1 + 0i$$

Fixed Point Analysis

Before



After



Fixed Point Analysis

I. Equations of Motion

$$z_1(t+dt) = z_1(t) + dt\frac{z_1(t) - z_2(t)}{|z_1(t) - z_2(t)|}$$

$$z_2(t+dt) = z_2(t) + dt\frac{z_2(t) - z_3(t)}{|z_2(t) - z_3(t)|}$$

$$z_3(t+dt) = \frac{z_3(t)}{z_2(t+dt)} + dt\frac{z_3(t) - z_4(t)}{z_2(t+dt)|z_3(t) - z_4(t)|} - \frac{z_1(t+dt)}{z_2(t+dt)}$$

$$z_4(t+dt) = \frac{z_4(t)}{z_2(t+dt)} + dt\frac{z_4(t) - z_1(t)}{z_2(t+dt)|z_4(t) - z_1(t)|} - \frac{z_1(t+dt)}{z_2(t+dt)}$$

Fixed Point Analysis

II. Calculate Time Derivatives

$$\dot{z}_i(t) = \lim_{dt \to \infty} \frac{z_i(t + dt) - z_i(t)}{dt}$$

III. Compute Jacobian

$$\left(\frac{\partial \dot{x}_i}{\partial x_j} \right)_{i,j=1,\ldots,4}$$

Local Analysis: Square

I. Eigenvalues

$$\lambda_1 = -i, \quad \lambda_2 = i, \quad \lambda_3 = \frac{-1}{2} - \frac{i\sqrt{7}}{2}, \quad \lambda_4 = \frac{-1}{2} + \frac{i\sqrt{7}}{2}$$

II. Eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \\ -i \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \\ i \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \frac{1}{4} - \frac{i\sqrt{7}}{4} \\ 1 \\ \frac{1}{4} - \frac{i\sqrt{7}}{4} \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} \frac{1}{4} + \frac{i\sqrt{7}}{4} \\ 1 \\ \frac{1}{4} + \frac{i\sqrt{7}}{4} \\ 1 \end{bmatrix}$$

Local Analysis: Line

I. Eigenvalues

$$\lambda_{1,2} = -2, \quad \lambda_3 = -1 - i, \quad \lambda_4 = -1 + i$$

II. Eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ i \\ 0 \\ 1 \end{bmatrix}$$

Local Analysis: Parallelogram

I. Eigenvalues

$$\lambda_1 = -i, \quad \lambda_2 = i, \quad \lambda_3 = \frac{-1}{2} - \frac{i\sqrt{7}}{2}, \quad \lambda_4 = \frac{-1}{2} + \frac{i\sqrt{7}}{2}$$

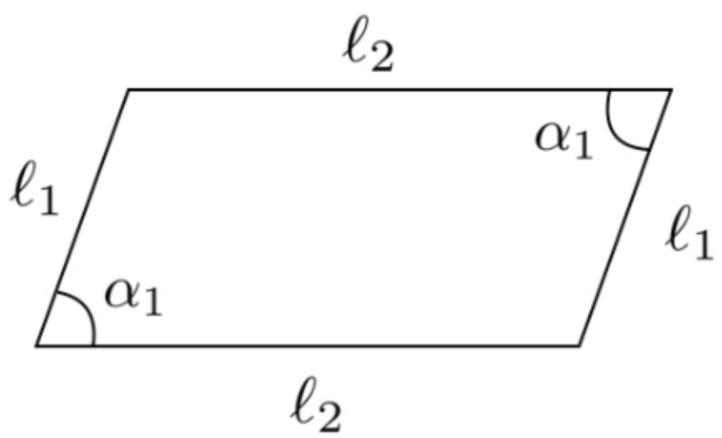
II. Eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \\ -i \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \\ i \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \frac{1}{4} - \frac{i\sqrt{7}}{2} \\ 1 \\ \frac{1}{4} - \frac{i\sqrt{7}}{2} \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} \frac{1}{4} + \frac{i\sqrt{7}}{2} \\ 0 \\ \frac{1}{4} + \frac{i\sqrt{7}}{2} \\ 0 \end{bmatrix}$$

Stability Analysis: Parallelogram

I. Change of Variables

$$x = \cos(\alpha_1) \quad \text{and} \quad s = \frac{\ell_1 - \ell_2}{\ell_1 + \ell_2}$$

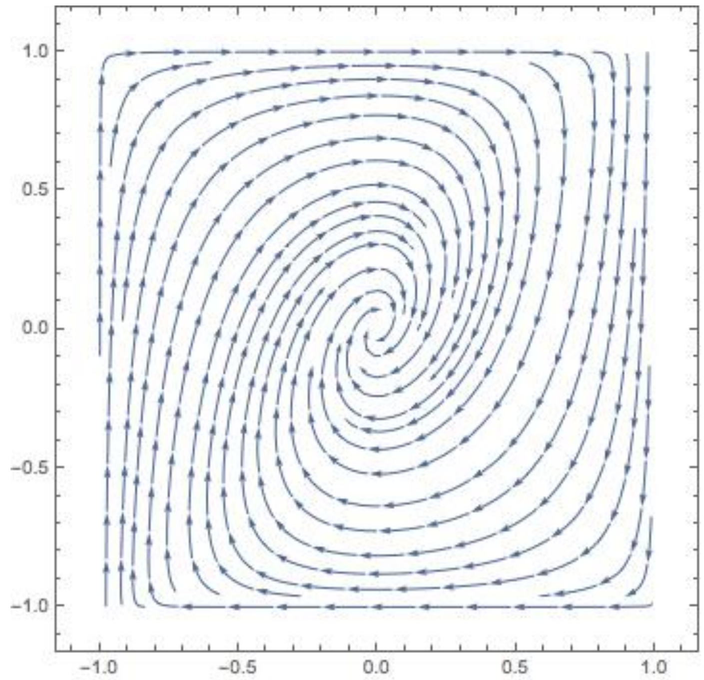


II. System of ODEs

$$\frac{dx}{d\tau} = -4s\frac{1 - x^2}{1 - s^2} \quad \text{and} \quad \frac{ds}{d\tau} = 2(x - s)$$

Stability Analysis: Parallelogram

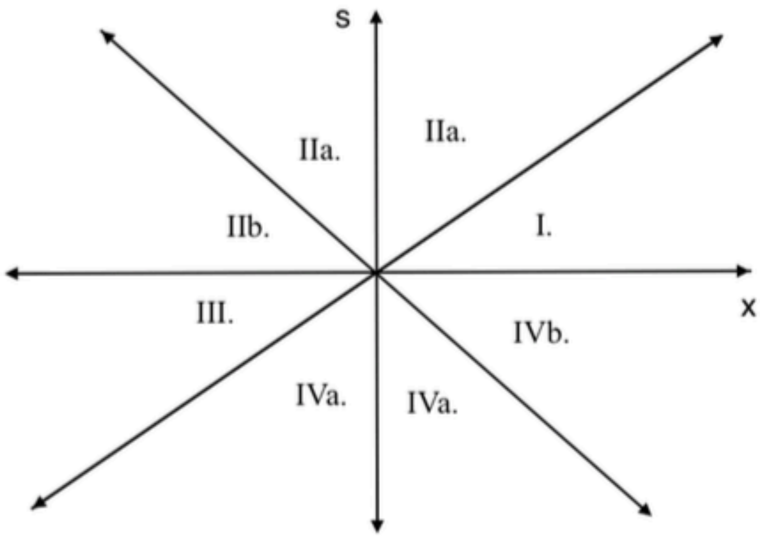
Theorem 1.2: *All configurations in the plane of parallelograms converge to a square*



Proof:

I. Partition Plane of Parallelogram

$$\begin{cases} \text{I.} & x > s > 0 \\ \text{IIa.} & s > 0, s > |x| \\ \text{IIb.} & s > 0, |x| > s, x < 0 \\ \text{III.} & x < s < 0 \\ \text{IVa.} & s < 0, |x| < |s| \\ \text{IVb.} & s < 0, |s| < x \end{cases}$$



Proof:

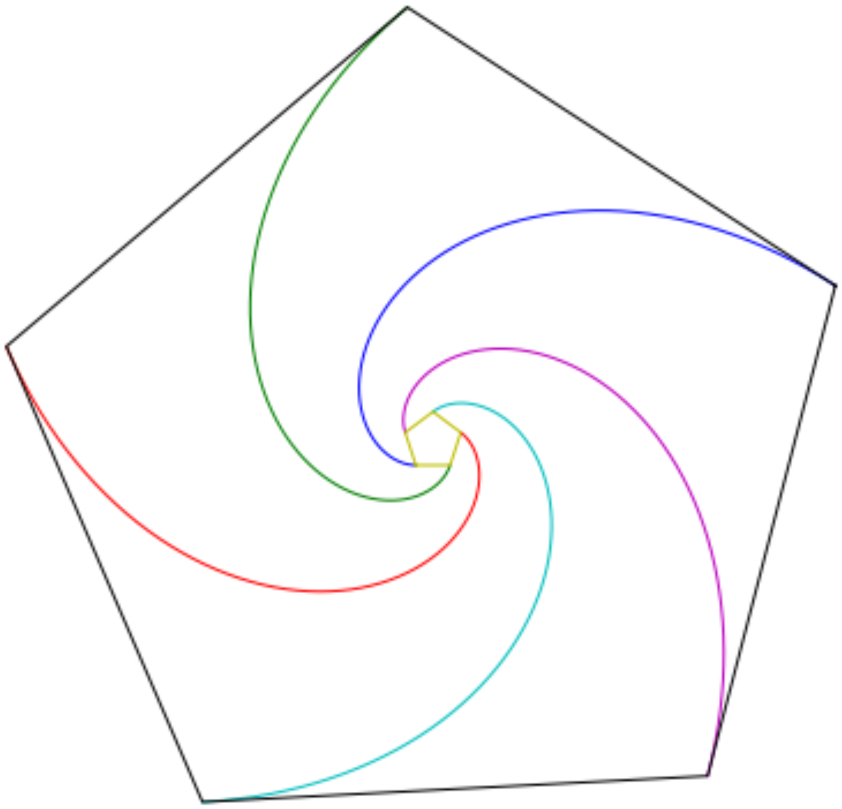
II. Bound our ODEs

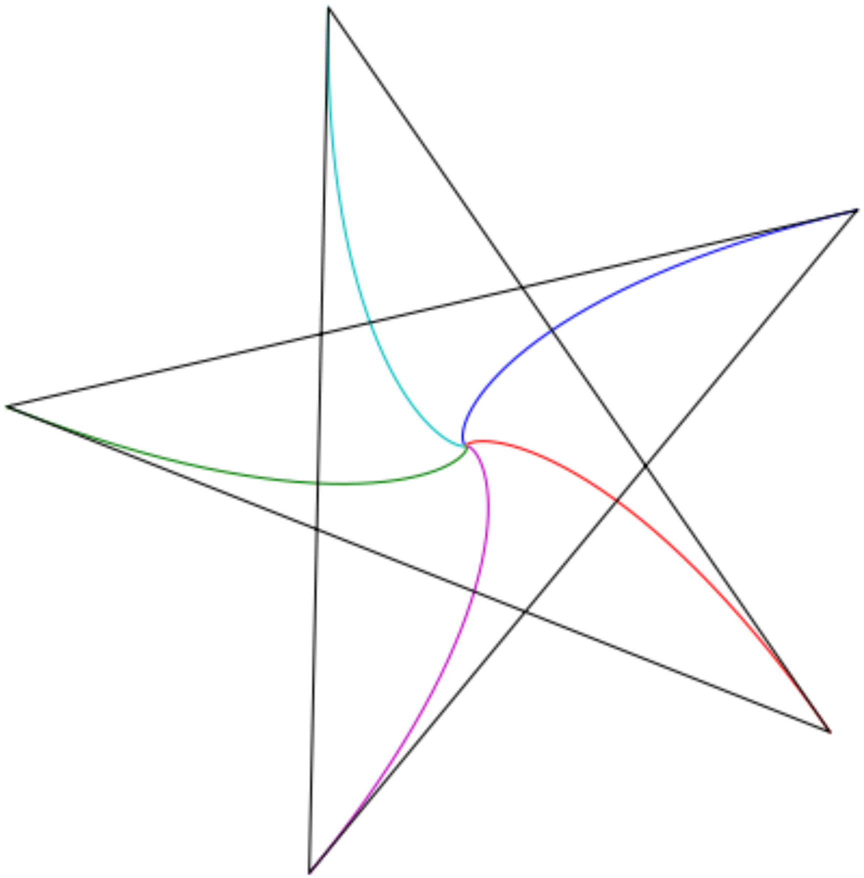
$$F_n(x,s) = \begin{cases} f_n(x,s) \\ 2(x-s) \end{cases}, \text{ such that } f_n(x,s) > -4s\frac{1-x^2}{1-s^2}$$

III. Compute Jacobian and
Eigenvalues

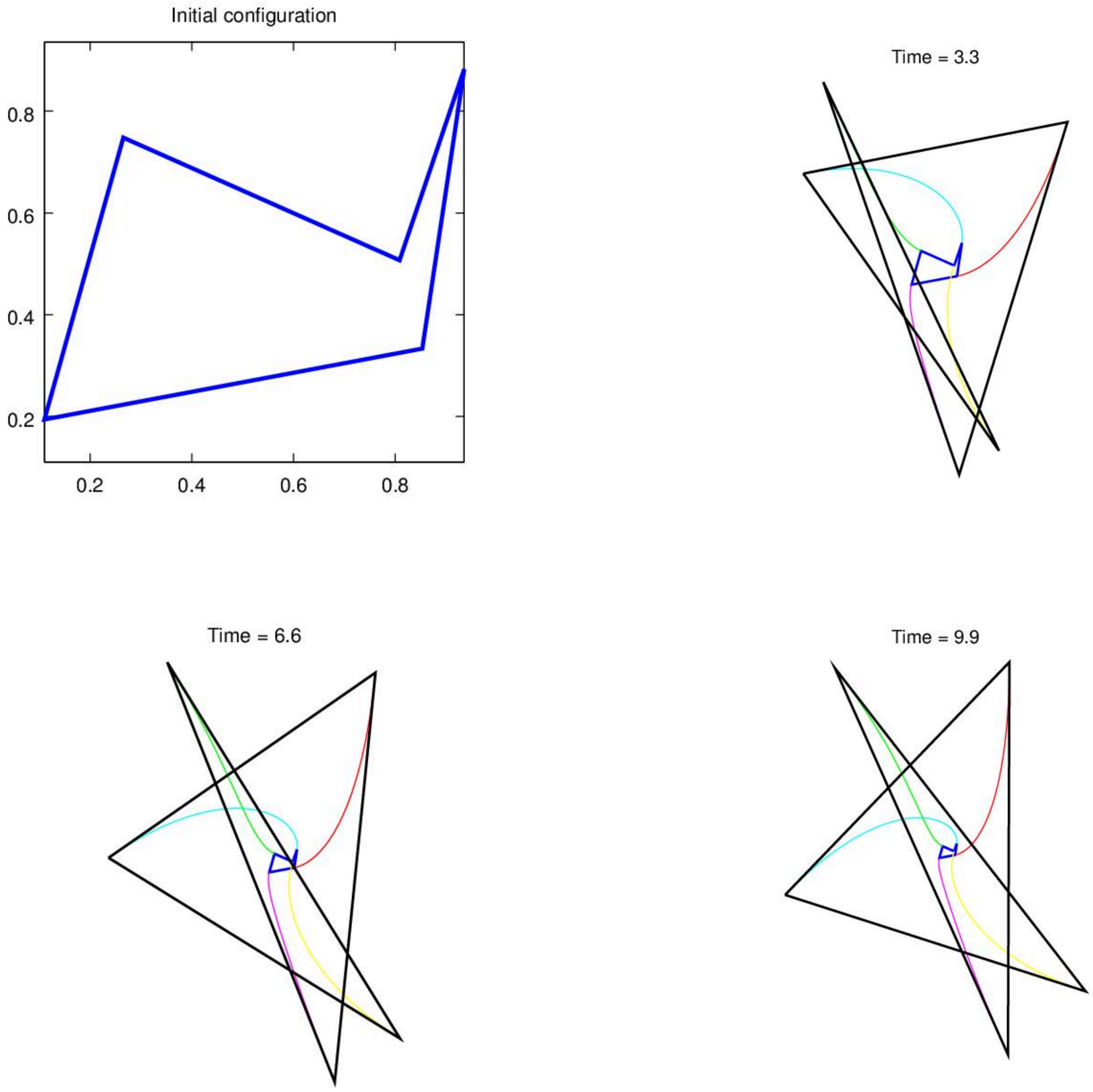
Normal Forms

Future Work: 5 Bugs





Experimentation



Experimentation