Discontinuous standard map dynamics

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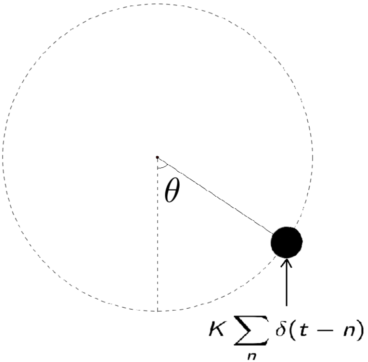
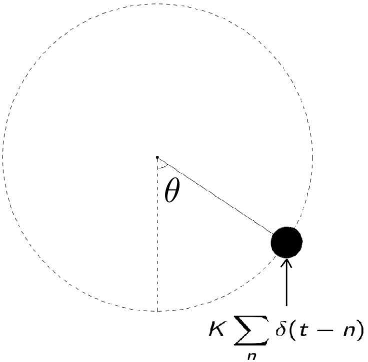


Consider a kicked rotator



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Integrate around $t = n$:

$$\theta_{n+1} = \theta_n + p_n \pmod{2\pi}$$
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Boundedness of momentum is understood using KAM theory.

Let's replace the sine wave by a square wave and shift horizontally:

$$
\begin{aligned}
x_{n+1} &= x_n + \alpha y_n \pmod 1 \\
y_{n+1} &= y_n + \operatorname{sgn}\left(x_{n+1} - \frac{1}{2}\right)
\end{aligned}
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KAM theory does not apply here because of the lack of smoothness.

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Understanding the dynamics of discontinuous maps of this kind could be of interest.

Which orbits are unbounded? Why do they appear to grow slowly?

Unbounded orbits?

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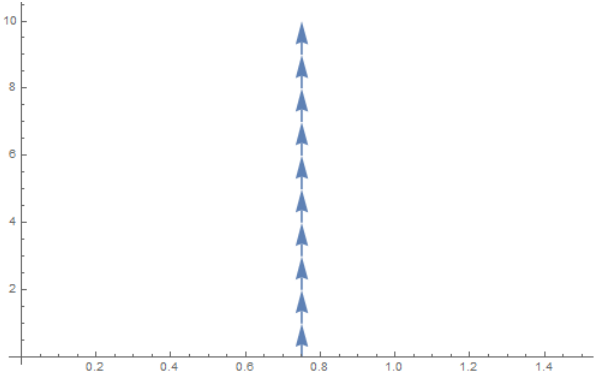

We'll restrict to $\alpha, y_0 \in \mathbb{Q}$. Reasons: More tractable, and maybe it'll give insight into the case $\alpha y_0 \notin \mathbb{Q}$.

For now, consider $\alpha = 1/q$ and $y_0 = a/b$.

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For convenience, expand the cylinder:

$$x' = x + \frac{y}{q} \pmod 1 \qquad\qquad x' = x + y \pmod q$$

$$y' = y + \operatorname{sgn}\left(x' - \frac{1}{2}\right) \qquad\qquad y' = y + \operatorname{sgn}\left(x' - \frac{q}{2}\right)$$

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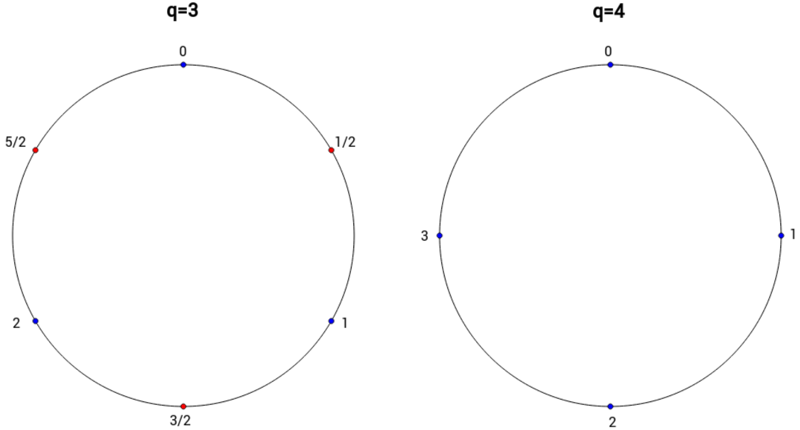
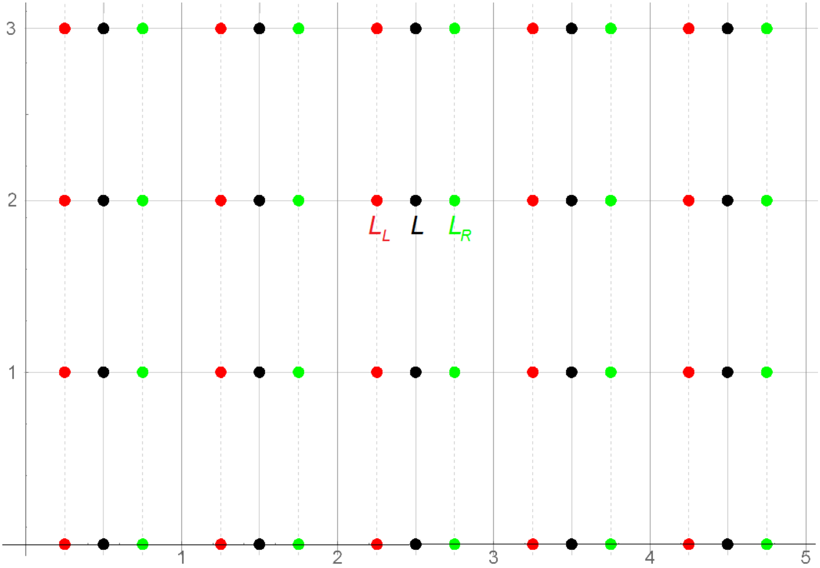
Can also define f on the torus – glue $y = 0$ to $y = q$.

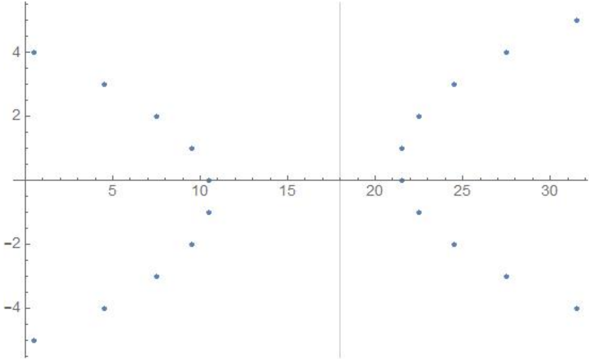
Invariant intervals ($\alpha = 1/q$ and $y_0 \in \mathbb{Z}$)

The action of f is the same throughout certain intervals.



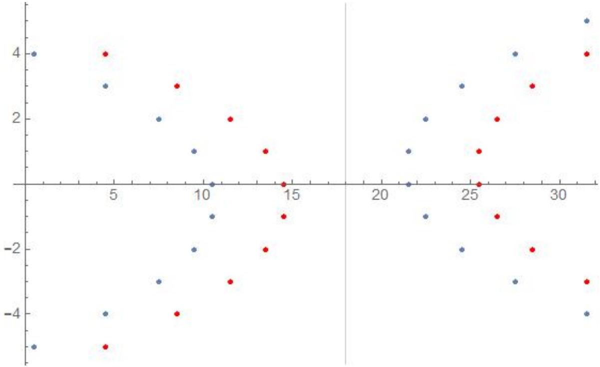

3
2
1
1
2
3
4
5
L_L
L
L_R

Rotational symmetry



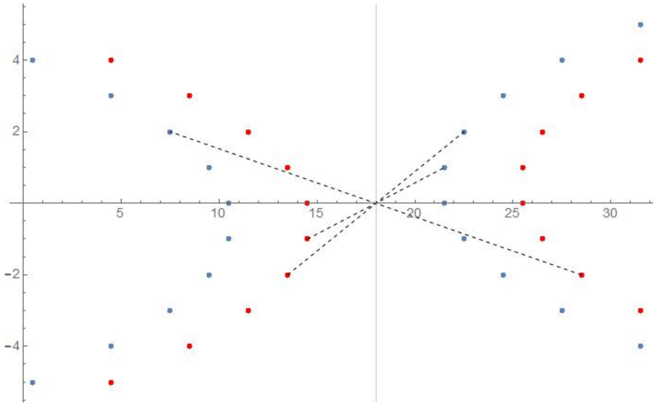
$$q = 36,\ x_0 = 18 + 3.5 = 21.5,\ y_0 = 0$$

Rotational symmetry



Red orbit: $x_0 = 18 - 3.5 = 14.5$

Rotational symmetry



Consider $y_0 \in \mathbb{Z}$ and $\alpha = 1/q$.

Proposition

Suppose q is even. For $(x_0, y_0) \in L$, the orbit under f is bounded.

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Proposition

Suppose q is odd. Then there exists a unique escaping orbit on L_R under f, and this orbit escapes to $+\infty$.

Can we find unbounded orbits for $\alpha = 1/q$ and $y_0 \in \mathbb{Q} \backslash \mathbb{Z}$?

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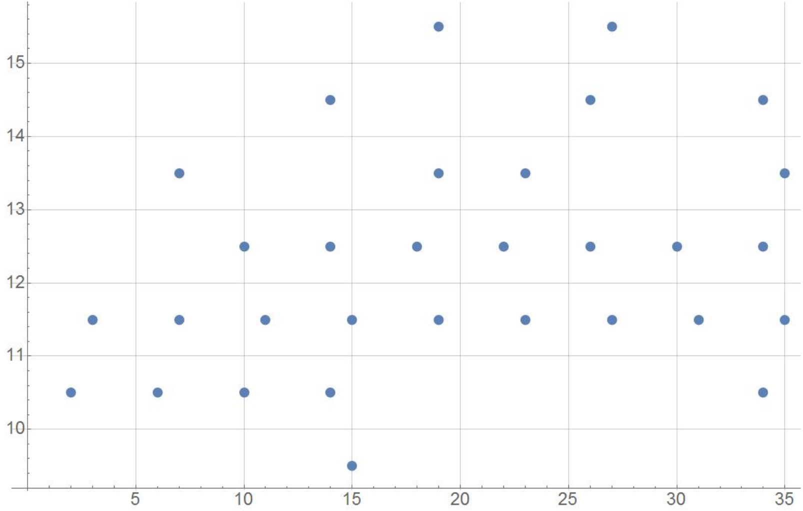
Consider points at level $y = \frac{q+1}{2} + m$.



30 iterates with $q = 18$

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Starting at $(r_0, y_0) = (q-3, \frac{q+1}{2})$, there are immediately two consecutive increases.

There is no point at level $y = \frac{q+1}{2} + 2$ from which there are two consecutive decreases.

Conjecture

For every $\alpha = 1/q$, there exists $y_0 = a/b$ such that there is an escaping orbit starting at some $(x_0, y_0) \in L_R$.

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We've shown this holds for q odd and $q \equiv 2 \pmod 4$.

Conjecture

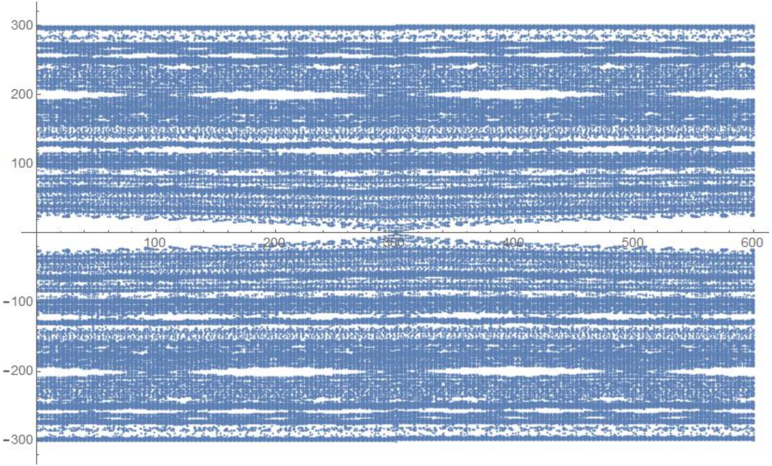
For every $\alpha = 1/q$, there exists $y_0 = a/b$ such that there is an escaping orbit starting at some $(x_0, y_0) \in L_R$.

We've shown this holds for q odd and $q \equiv 2 \pmod 4$.

q	24	28	32	36	40	44	48	52	56	60
a	36	67	63	77	19	23	360	243	23	254
b	103	144	205	227	337	223	1043	1264	505	1379

For $q \equiv 0 \pmod 4$, this table gives $y_0 = a/b$ with the smallest b.

Islands at $y = q/n$ for n odd ($y_0 \in \mathbb{Z}$ again)

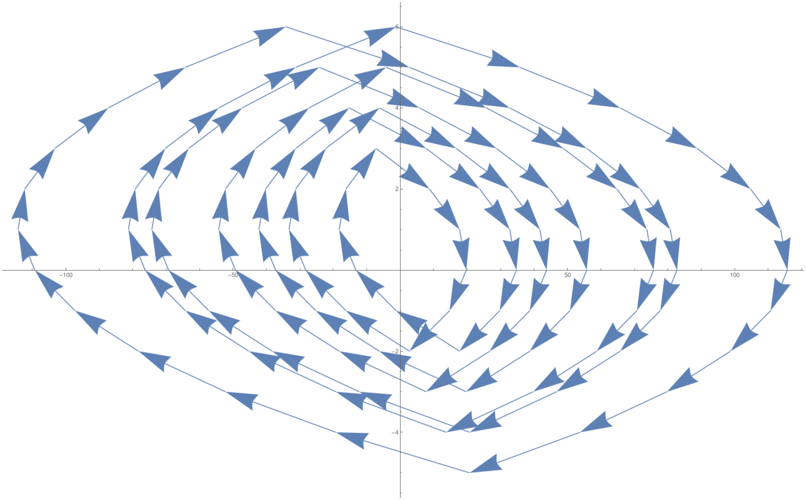


The escaping orbit for $q = 601$.

First return map:

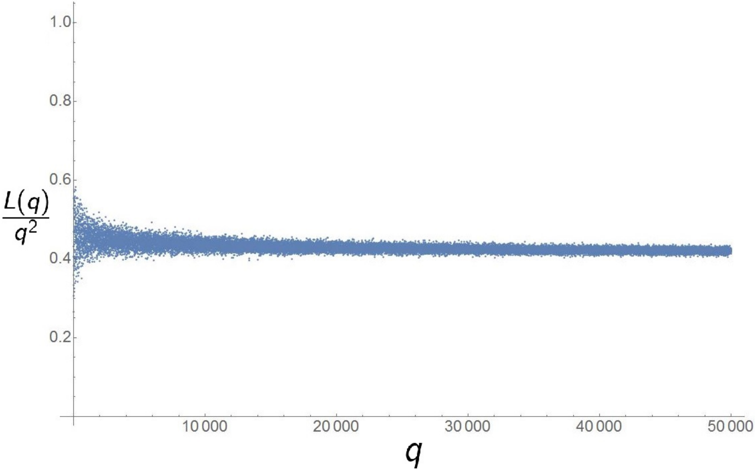
$$c_{m+1} = c_m + nd_m - \left(\frac{n-1}{2}\right)^2 - r$$
$$d_{m+1} = d_m - \operatorname{sgn}(c_{m+1})$$

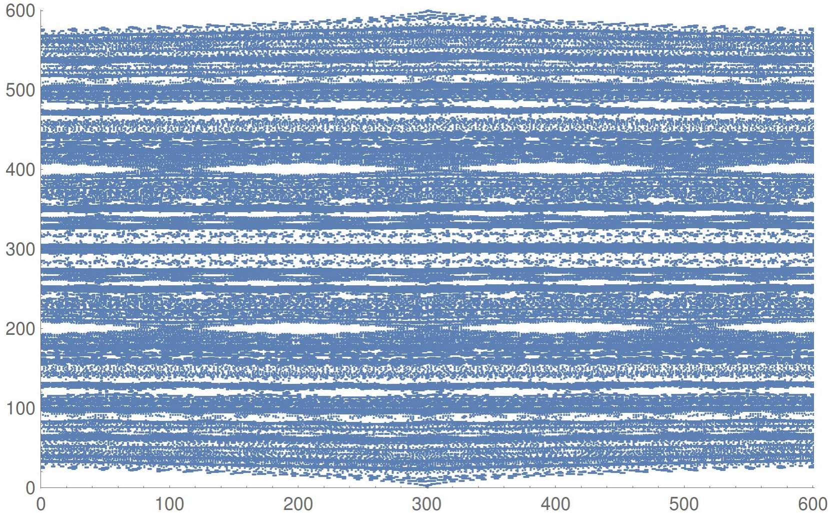
for all $m \geq 0$, where $r = q \pmod{n}$.



6
4
2
-2
-4
-100
-50
50
100

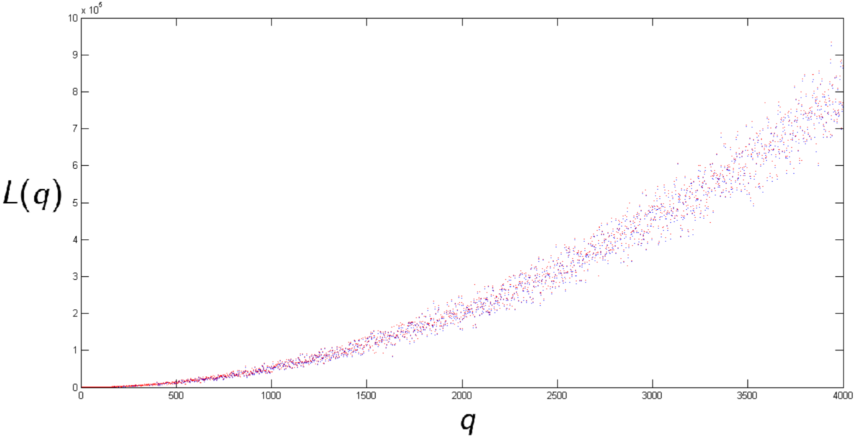
Asymptotics of the escaping orbit




600
500
400
300
200
100
0
0
100
200
300
400
500
600

$$L(q) > Cq \log q$$

for some constant $C > 0$ and sufficiently large q.


x 10^5
10
9
8
7
6
5
4
3
2
1
0
$L(q)$
0
500
1000
1500
2000
2500
3000
3500
4000
q

Acknowledgements and References

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M. Arnold and V. Zharnitsky. *Communications in Mathematical Physics* **338**, 501-521 (2015).

S. M. Ulam. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Contributions to Astronomy, Meteorology, and Physics, 315–320, University of California Press, Berkeley, Calif., 1961. http://projecteuclid.org/euclid.bsmsp/1200512818.