Mixing Time in Robotic Explorations

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Outline

1. Motivation
2. Model and Definitions
3. A Simple Room Example
4. Rooms
   - Comb Room and Snake Room
   - A Lego Room
   - A General Room Example
5. Tunnel
   - Tilted Tunnel
   - Bent Tunnel
6. References

Motivation

- Roomba

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Model and Definitions

Room

$\{A_j = [a_j, b_j] \times [c_j, d_j]\}_{j=1}^{n} \setminus \partial A \setminus w$

$w := \{w_1, w_2, \cdots\}$ is the set of interior walls.

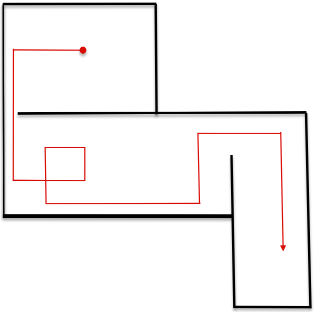
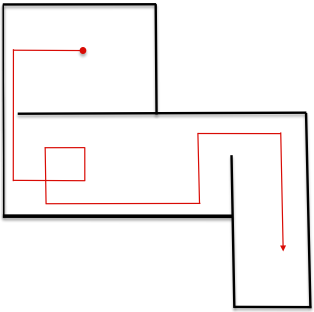


Figure 1: Possible paths taken by the point robot

Motion of the point robot

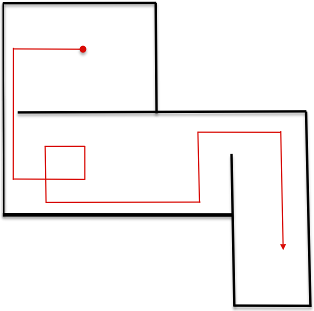
- Horizontal move h_i and Vertical Move h_i



Model and Definitions

Motion of the point robot

- Horizontal move h_i and Vertical Move v_i.
- Step: an ordered pair of moves (h_i, v_i).



Definition of regions

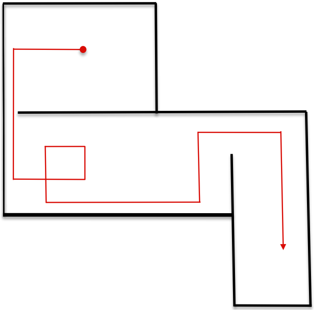


Figure 2: Possible paths of robot starting from the red circle

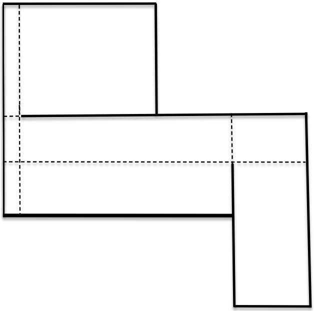
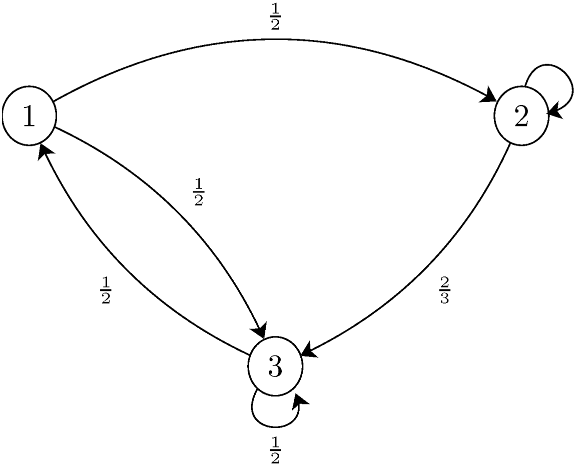


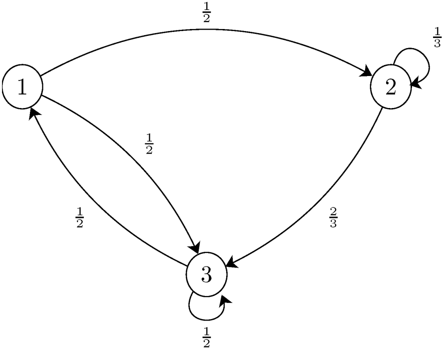
Figure 3: Definition of regions in a typical room configuration

Definition (Markov Chain)

A finite Markov Chain is a process which moves among the elements of a finite set Ω so that when at $x \in \Omega$, the next state is chosen according to a fixed probability distribution $P(x, \cdot)$.



$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix}
\end{array}
$$



Definition (Transition Matrix)

The matrix P that that represents the Markov process with state space Ω is called the transition matrix. P is stochastic. That is, for all x^{th} row of P, $P(x, \cdot)$ satisfies:

$$\sum_{y \in \Omega} P(x, y) = 1 \tag{1}$$

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$$\sum_{y \in \Omega} P(x, y) = 1 \tag{1}$$

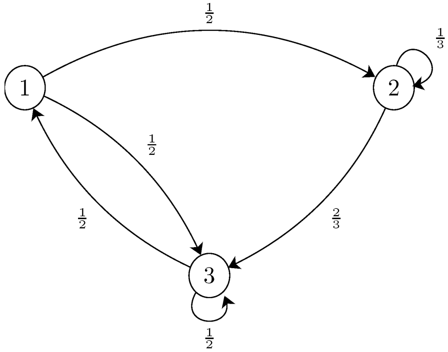
Theorem

Every eigenvalue λ of a stochastic matrix P satisfies $|\lambda| \leq 1$.

Definition (Stationary Distribution)

A *stationary distribution* π on Ω satisfies:

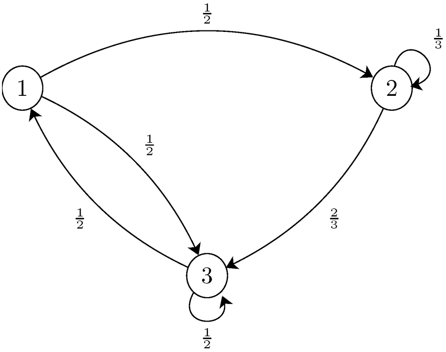
$$\pi = \pi P \tag{2}$$



Model and Definitions

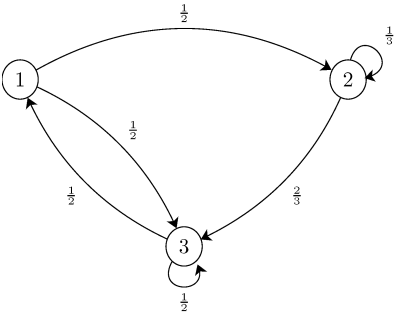
- **Irreducibility** :

A transition matrix P is irreducible if $\forall x, y \in \Omega$, there exists integer t such that $P^t(x, y) > 0$.



- **Aperiodicity**:

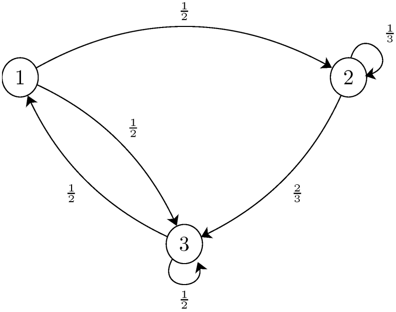
Period is the greatest common divisor of $\tau(x) := \{t \geq 1 : P^t(x,x) > 0\}$. A transition matrix P is aperiodic if all states have period 1.



- **Reversibility**:

A transition matrix is *reversible* if it satisfies:

$$\pi(x)P(x,y) = \pi(y)P(y,x) \quad \text{for all} \quad x, y \in \Omega \tag{3}$$



Definition

The total variation distance (TV) between two probability distribution μ and v on Ω is defined as the maximum difference between the probabilities assigned to a single event by the two distributions:

$$||\mu - v||_{TV} = \max_{A \subset \Omega} |\mu(A) - v(A)| \tag{4}$$

Theorem (Convergence Theorem)

Suppose that P is irreducible and aperiodic, with stationary distribution π. For all t, there exists constants $\alpha \in (0,1)$ and $C > 0$ such that:

$$\max_{x\in\Omega} ||P^t(x,\cdot) - \pi||_{TV} \leq C\alpha^t \tag{5}$$

Model and Definitions

Definition (Mixing Time)

Let $d(t) := \max_{x \in \Omega} ||P^t(x, \cdot) - \pi||_{TV}$, then the mixing time t_{mix} is defined by:

$$t_{mix}(\delta) := min\{t : d(t) \leq \delta\} \tag{6}$$

Model and Definitions

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Choose $\delta = 1/100$, and

$$t_{mix} := t_{mix}(1/100)$$

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A Simple Room

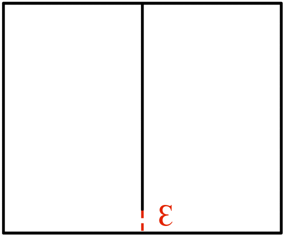


Figure 4: A simple Room

A Simple Room

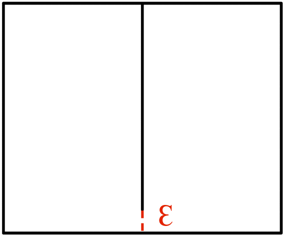


Figure 4: A simple Room

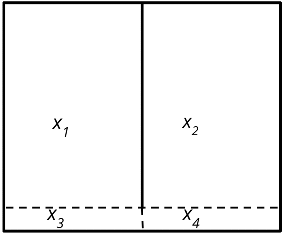
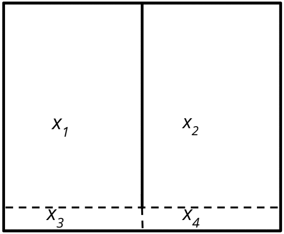


Figure 5: Labeled regions

A Simple Room



$$P = \begin{array}{c|c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 1-\epsilon & 0 & \epsilon & 0 \\ 0 & 1-\epsilon & 0 & \epsilon \\ \frac{1}{2}(1-\epsilon) & \frac{1}{2}(1-\epsilon) & \frac{1}{2}\epsilon & \frac{1}{2}\epsilon \\ \frac{1}{2}(1-\epsilon) & \frac{1}{2}(1-\epsilon) & \frac{1}{2}\epsilon & \frac{1}{2}\epsilon \end{pmatrix} \end{array}.$$

A Simple Room

Relaxation time t_{rel}

- P is a reversible and stochastic, so we can label its eigenvalues in descending order:

$$1 = |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_{|\Omega|}| \geq -1 \tag{7}$$

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- Spectral gap of P is $\gamma := 1 - |\lambda_2|$

A Simple Room

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- Spectral gap of P is $\gamma := 1 - |\lambda_2|$

Definition (Relexation Time)

The *relaxation time* t_{rel} of P with spectral gap γ is defined as:

$$t_{rel} := \frac{1}{\gamma} \tag{8}$$

A Simple Room

Relation between t_{mix} and t_{rel}:

Theorem

Let $\pi_{min} := \min_{x\in\Omega} \pi(x)$. For a reversible, irreducible and aperiodic Markov chain with state space Ω , the relation between its relaxation time t_{rel} and π_{min} can be represented as:

$$\log(\frac{1}{\delta\pi_{min}})t_{rel} \geq t_{mix}(\delta) \geq (t_{rel} - 1)\log(\frac{1}{2\delta}) \tag{9}$$

A Simple Room

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$$\log(\frac{1}{\delta\pi_{min}})t_{rel} \geq t_{mix}(\delta) \geq (t_{rel} - 1)\log(\frac{1}{2\delta}) \tag{9}$$

Therefore, t_{mix} and t_{rel} are on the same order.

A Simple Room

- Computation Results: $|\lambda_2| = 1 - \epsilon$
  $t_{mix} = 1/\gamma = 1/(1-\epsilon) = \Theta(\frac{1}{\epsilon})$.

$$P = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 1-\epsilon & 0 & \epsilon & 0 \\ 0 & 1-\epsilon & 0 & \epsilon \\ \frac{1}{2}(1-\epsilon) & \frac{1}{2}(1-\epsilon) & \frac{1}{2}\epsilon & \frac{1}{2}\epsilon \\ \frac{1}{2}(1-\epsilon) & \frac{1}{2}(1-\epsilon) & \frac{1}{2}\epsilon & \frac{1}{2}\epsilon \end{pmatrix} \end{array}.$$

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- Simulation Results:

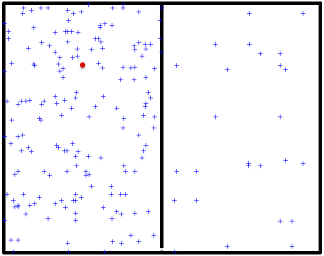

Figure 6: Simulation Results
$n = 100$ and $\epsilon = 0.001$

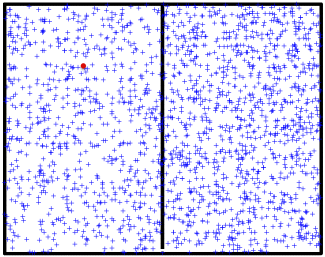

Figure 7: Simulation Results
$n = 1000$ and $\epsilon = 0.001$

Proposition

Horizontal (vertical) scaling does not change t_{mix}.

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Definition (Bottleneck Ratio)

After scaling the room to unit dimensions, we define the length of the smallest horizontal (vertical) gap as ϵ, which is also the bottleneck ratio.

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Comb Room

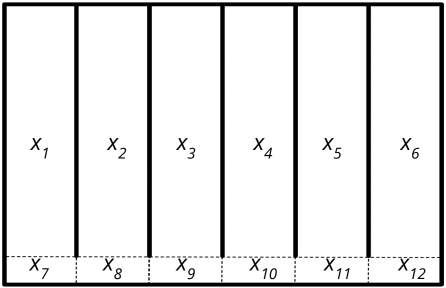


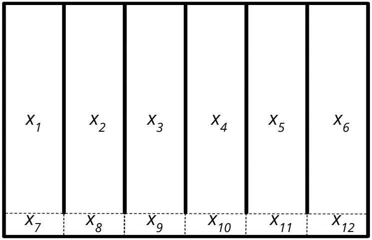
Figure 8: A "Comb" Shape Room With $N = 6$

Comb Room: Matrix Approach

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$P_{11} = (1-\epsilon)I$,
$P_{12} = \epsilon I$,
$P_{21} = \frac{1-\epsilon}{N} J$
$P_{22} = \frac{\epsilon}{N} J$.
I is the $N \times N$ identity matrix,
and J is the $N \times N$ matrix with all entries being one.

Comb Room



- $|\lambda_2| = 1 - \epsilon$
- $t_{mix} = \Theta(1/\epsilon)$

Snake Room (ouroboric)

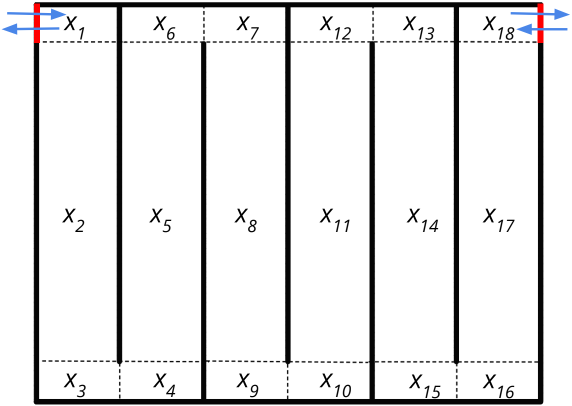
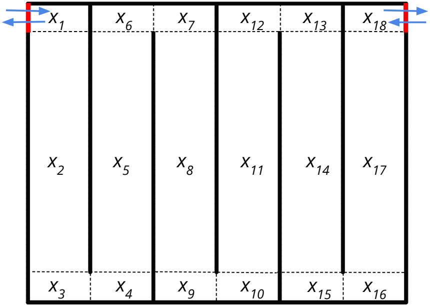


Figure 9: An Ouroboric Snake Shape Room With $N = 6$

Ouroboric Snake

Figure 10: An Ouroboric Snake



$$
\begin{array}{c c}
 & \begin{array}{ccccccccc} x_{3n-5} & x_{3n-4} & x_{3n-3} & x_{3n-2} & x_{3n-1} & x_{3n} & x_{3n+1} & x_{3n+2} & x_{3n+3} \end{array} \\
\begin{array}{c} x_{3n-2} \\ x_{3n-1} \\ x_{3n} \end{array} &
\begin{pmatrix}
\frac{\epsilon}{2} & \frac{1-2\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{1-2\epsilon}{2} & \frac{\epsilon}{2} & & & \\
 & & & \epsilon & 1-2\epsilon & \epsilon & & & \\
 & & & \frac{\epsilon}{2} & \frac{1-2\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{1-2\epsilon}{2} & \frac{\epsilon}{2}
\end{pmatrix}
\end{array}
$$

The k^{th} eigenvectors r_k has the form:

$$r_k = \begin{bmatrix} a \\ b \\ c \\ ae^{2\pi ik/N} \\ be^{-2\pi ik/N} \\ ce^{-2\pi ik/N} \\ \vdots \\ ae^{-2\pi ik(N-1)/N} \\ be^{-2\pi ik(N-1)/N} \\ ce^{-2\pi ik(N-1)/N} \end{bmatrix}$$

where $k = 0, 1, 2, \cdots, N-1$ and a, b, c are three constants depending on N and k.

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where $k = 0, 1, 2, \cdots, N-1$ and a, b, c are three constants depending on N and k.

$$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \epsilon/2(1+e^{\frac{2\pi ik}{N}}) & (1/2-\epsilon)(1+e^{\frac{2\pi ik}{N}}) & \epsilon/2(1+e^{\frac{2\pi ik}{N}}) \\ \epsilon & 1-2\epsilon & \epsilon \\ \epsilon/2(1+e^{\frac{-2\pi ik}{N}}) & (1/2-\epsilon)(1+e^{\frac{-2\pi ik}{N}}) & \epsilon/2(1+e^{\frac{-2\pi ik}{N}}) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- Trace: $1 - \epsilon + \epsilon \cos(\frac{2\pi k}{N})$
- When $k = 1$,

$$t_{mix} = \frac{1}{\epsilon(1 - \cos(\frac{2\pi k}{N}))} \approx \frac{N^2}{2\pi^2 \epsilon}$$

which is of $\Theta(N^2/\epsilon)$.

Non-ouroboric Snake

- Shape

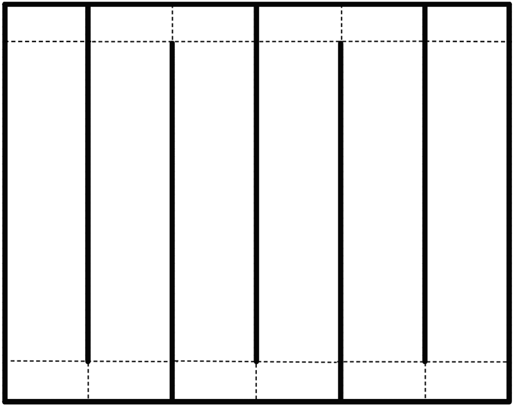

Figure 11: non-ouroboric snake shape

Non-ouroboric Snake

- Coupling Method
    - Definitions

Definition (Coupling of Markov Chains)

A *coupling of Markov chains* with transition matrix P is a process $(X_t, Y_t)_{t=0}^{\infty}$ with the property that both (X_t) and (Y_t) are Markov chains with transition matrix P, although the two chains may have different starting distribution.

Definition (t_{coup})

The coupling time $t_{coup} := \min\{t : X_t = Y_t\}$

Non-ouroboric Snake

- Coupling Method
  - How to bound t_{mix}

Theorem

Suppose that for each pair of states $x, y \in \Omega$ there is a coupling (X_t, Y_t) with $X_0 = x$ and $Y_0 = y$. Then, for each such coupling,

$$d(t) \leq \max_{x,y \in \Omega} P_{x,y}\{t_{coup} > t\} \tag{10}$$

Theorem (Markov's Inequality)

If X is any nonnegative random variable and $a > 0$, then

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}. \tag{11}$$

Corollary

$$t_{mix} \leq 100 E_{x,y}(t_{coup}) \tag{12}$$

Non-ouroboric Snake

- Coupling Method
  - Design a coupling

Definition (Specific coupling design for this case)

For any two points x, y, at each step, let x move first and then y move. At each step, y always moves to the same vertical height as x. If x and y are in the same chamber, then y also moves to the same horizontal location as x.

Theorem (Observation)

$E_{x,y}(t_{coup})$, in this case, is bounded above by the expected time for one point to move from the first chamber to the last chamber.

Non-ouroboric Snake

- Redefine States

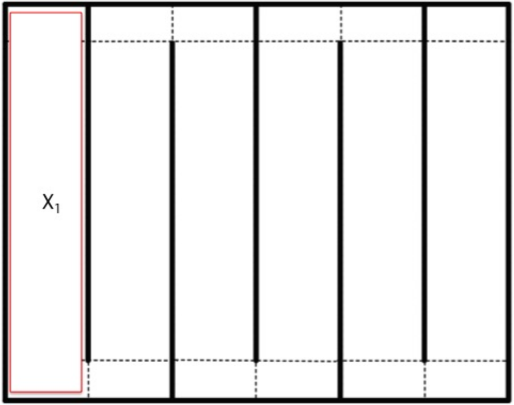

Figure 12: Simplified States

Non-ouroboric Snake

- Random Walk On A Graph

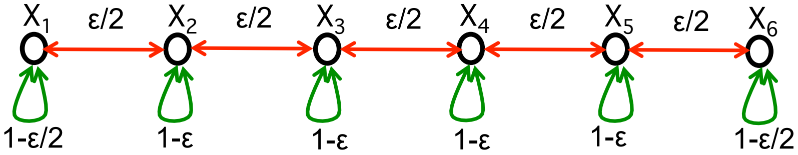

Figure 13: Simplified Random Walk On A Graph

Non-ouroboric Snake

- Solve expected time from x_1 to x_N: If we denote the expected time of moving from the n^{th} chamber to the last chamber (the N^{th}) as $T(n)$, then we would easily obtain a following recurrence relation:

$$T(n+1) - 2T(n) + T(n-1) + 1/\epsilon = 0 \tag{13}$$

with boundary conditions:

$$T(0) = T(1) + 2/\epsilon, \quad T(N) = 0 \tag{14}$$

Non-ouroboric Snake

- Bound mixing time t_{mix}: After solving this relation, we find that

$$T(n) = -\frac{3n}{2\epsilon} - \frac{n^2}{2\epsilon} + \frac{3N}{2\epsilon} + \frac{N^2}{2\epsilon} \tag{15}$$

  Therefore we would have
  $t_{mix} \leq 100 \cdot E(t_{coup}) \leq 100 \cdot T(0) = \frac{150N}{\epsilon} + \frac{50N^2}{\epsilon}$. Therefore, we know that the mixing time t_{mix} in this case is also bounded above by $O(\frac{N^2}{\epsilon})$.

A Lego Room

Definition

A room is a n-Lego room if and only if it consists of n unit chambers and each chamber is connected to at least one other chamber. The walls between any two connected chamber is of length $1 - \epsilon$.

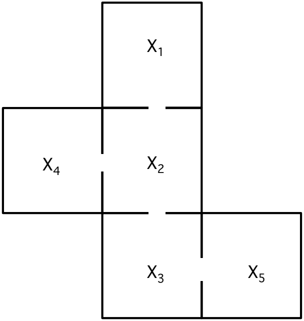

Figure 14: An Example of 5-Lego Room

A Lego Room

- Random Walk

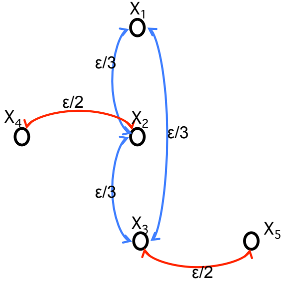

Figure 15: The Equivalent Random Walk On a Graph

A Lego Room

Theorem (The Wall Theorem)

The mixing time t_{mix} for a room increases when the length of one wall is extended and decreases when it is shortened.

Corollary (Special Case Of The Wall Theorem)

For any random walk on a graph G, if the probability between state i and state j is decreased (the probability of staying in i and j is increased), then the mixing time t_{mix} for this process increases. If such probability is increased, then t_{mix} decreases.

A Lego Room

Transformation by the previous Corollary:

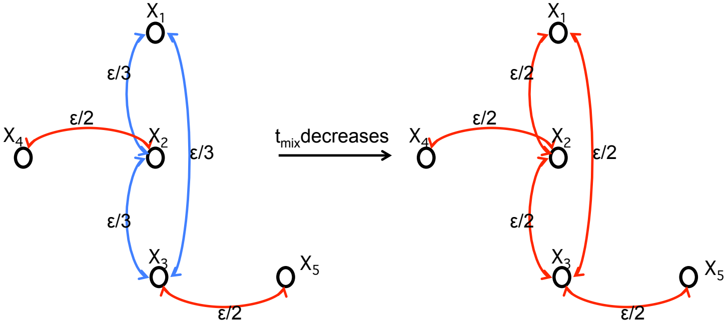

Figure 16: A transformation that decreases mixing time

A Lego Room

Transformation by TWT and its Corollary:

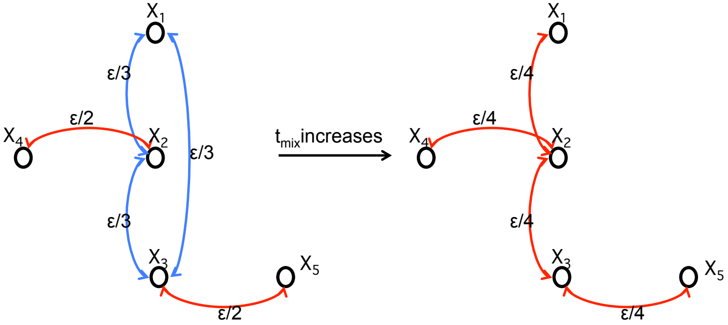

Figure 17: A transformation that increases mixing time

A Lego Room

Definition

A red random walk on a graph G is a random walk such that the probability from any vertex i to vertex j of G (in one step) is either 0 or $q\epsilon$, where q is a constant for this walk.

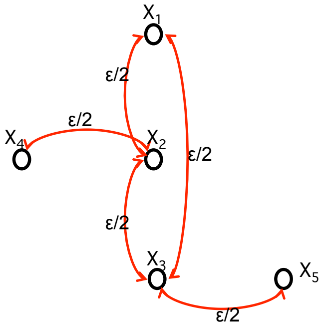

Figure 18: Transformation 1

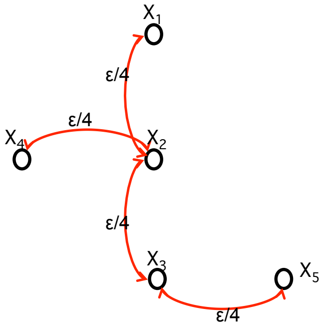

Figure 19: Transformation 2

A Lego Room

Definition (Laplacian Matrix)

Let $G = (V, E)$ be a non-directed finite graph. Let V be the set of vertices and $|V| = N$. Then after choosing a fixed ordering $w_1, w_2, ..., w_N$ of the set V, the *Laplacian matrix* is the N by N matrix $A(G)$ whose diagonal entries a_{ii} being the valencies of vertex i and off diagonal entries $a_{ij} = a_{ji} = -1$ if vertex i and j are connected and 0 otherwise.

Definition (Algebraic Connectivity)

Let $n \geq 2$ and $0 \leq \lambda_1 \leq \lambda_2 = a(G) \leq \lambda_3 \leq \cdots \leq \lambda_n$ be the eigenvalues of the matrix $A(G)$. The *algebraic connectivity* of the graph G is the second smallest eigenvalue $a(G)$.

A Lego Room

Theorem (Fiedler, 1973)

Denote $e(G)$ as the edge connectivity of a connected graph G, which is the minimal number of edges whose removal would result in losing connectivity of the graph G. Then for any G, we have

$$N \geq a(G) \geq e(G)(1 - \cos(\pi/N)) \tag{16}$$

Notice that the second largest eigenvalue of transition matrix P for a red random walk on G is $\lambda_2 = 1 - q\epsilon a(G)$.

Theorem (Mixing Time for A Lego Room)

If a room is a N-Lego room, then the mixing time t_{mix} for this room is bounded below from $O(\frac{1}{N\epsilon})$ and bounded above by $O(\frac{N^2}{\epsilon})$.

A General Room

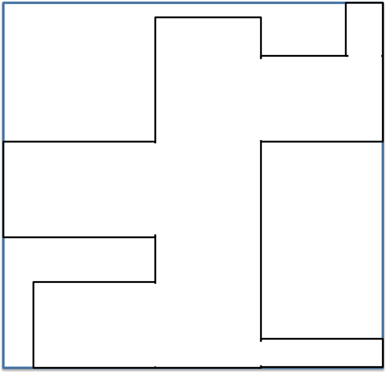

Figure 20: A Room

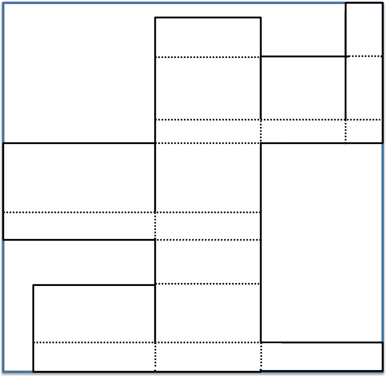

Figure 21: Adding Walls

A General Room

Lemma

For any room, the number of states is on the order of $O(s)$, where s is the number of sides.

Lemma

The probability between any two connected states is bigger than or equal to ϵ.

Then by TWT, we can decrease the probability from any state i to any other state j to ϵ with t_{mix} increasing. Therefore, t_{mix} for the original room is bounded by t_{mix} for a red random walk.

Theorem

For any room with s many number of sides and ϵ bottleneck ratio, the mixing time t_{mix} is bounded above by $O(\frac{s^2}{\epsilon})$

Outline

1 Motivation

2 Model and Definitions

3 A Simple Room Example

4 Rooms
- Comb Room and Snake Room
- A Lego Room
- A General Room Example

5 Tunnel
- Tilted Tunnel
- Bent Tunnel

6 References

Tilted Tunnel

- Shape

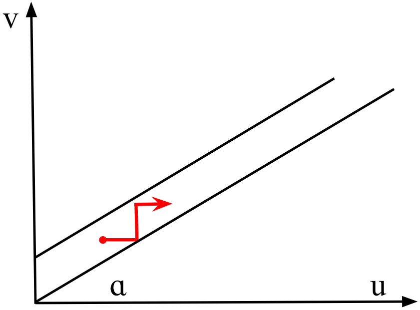

Figure 22: A Tilted Tunnel

Tilted Tunnel

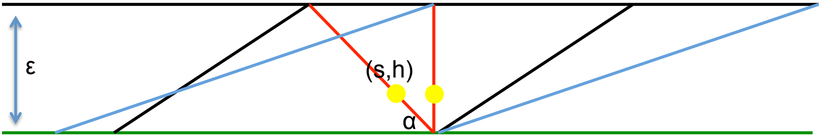

Figure 23: A Tilted Tunnel

$$P\rho(s,h,t) = \rho(s,h,t+1) = \frac{1}{B}\iint_{D_B} \rho(u,r,t)\,dudr \tag{17}$$

$$\rho(s,h,t+1) = \sum_{k=0}^{\infty} a_k(t+1)e^{2\pi iks/L} = \frac{1}{B}\iint_{D_B} \rho(u,r,t)\,dudr$$

$$= \frac{1}{B}\iint_{D_B} \sum_{k=0}^{\infty} a_k(t)e^{2\pi iku/L}dudr = \frac{1}{B}\sum_{k=0}^{\infty}\iint_{D_B} a_k(t)e^{2\pi iku/L}dudr$$

Tilted Tunnel

$$a_k(t+1)e^{2\pi iks/L} = \frac{1}{B}\iint_{D_B} a_k(t)e^{2\pi iku/L}dudr \tag{18}$$

$$a_k(t+1) = \frac{L^2\sin^2(2\alpha)}{4\epsilon^2k^2\pi^2}\sin\left(\frac{2\epsilon k\pi}{L\sin(2\alpha)}\right)a_k(t) = \Phi(k)a_k(t) \tag{19}$$

where $\Phi(k)$ is the eigenvalues in this case. When $k = 1$, such value is the second largest.

Theorem (Mixing Time For Tilted Tunnel)

For a Tunnel of length L and width ϵ, where $\epsilon \ll L$, the mixing time t_{mix} is on the order of $O(\frac{\sin^2(2\alpha)L^2}{\epsilon^2})$

Bent Tunnel

Conjecture

For any bent tunnel L with width ϵ, where $\epsilon \ll L$, we denote $\alpha(s)$ as the angle of the tunnel with horizontal axis at point s. Then

$$t_{rel} = \frac{3}{4\pi^2\epsilon^2}\left(\int_L |\sin(2\alpha(s))|ds\right)^2 \tag{20}$$

Experimentation:

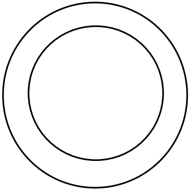

Figure 24: An Example

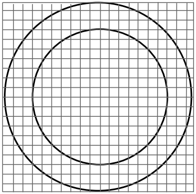

Figure 25: Discretization

Bent Tunnel

Some data: 90 by 90 pixels discretization

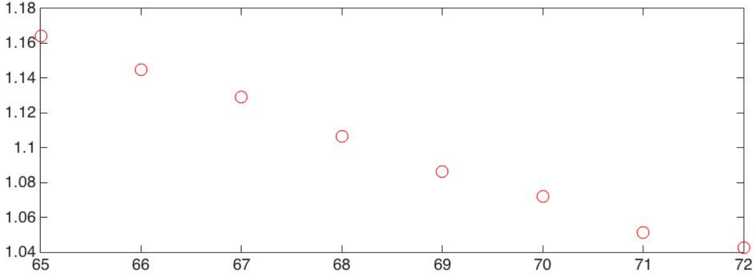

Figure 26: Expected Result to Discretizated Result

Outline

1. Motivation
2. Model and Definitions
3. A Simple Room Example
4. Rooms
   - Comb Room and Snake Room
   - A Lego Room
   - A General Room Example
5. Tunnel
   - Tilted Tunnel
   - Bent Tunnel
6. References

References

- David A. Levin, Yuval Peres, and Elizabeth L. Wilmer, *Markov Chains and Mixing Time*, American Mathematical Soc, 2009.
- Miroslav Fiedler, *Algebraic Connectivity of Graphs*, Czechoslovak Math. J. 23(98): 298 - 305, 1973.
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- Many thanks to Yuliy Baryshnikov, Maxim Arnold and Stefan Klajbor Goderich.