Singularities of Hinge Structures

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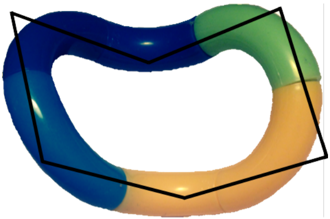
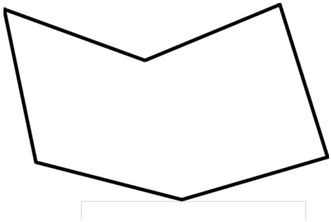
Origin

Hinge Structures

Introduction
Vector Fields
Fibrations
Javelins





Definition

A **hinge structure** is a piecewise linear embedding of $I = [0, 1]$ into $\mathbb{R}^3$. We call the hinge vectors $v_0, v_1, \ldots, v_{k-1}$ and the vertices $p_0, p_1, \ldots, p_k$ ($v_i = p_{i+1} - p_i$). In a closed configuration, $p_0 = p_k$. (For closed structures, we have an embedding of S^1.)

Singularities

The angles between consecutive hinge vectors are fixed, but the hinges are free to rotate.

The set of positions is parametrized by the $k-1$-dimensional torus, T^{k-1}, and we have a map $\tau : T^{k-1} \longrightarrow W_1(\mathbb{R}^3)$, which gives the position of an end-frame that we attach to p_{k-1}.

We are looking for configurations $x \in T^{k-1}$ where τ has a singularity. In other words, we want to find all the positions where the motion of τ is restricted.

Definition

A hinge structure is **singular** in a given configuration x if τ is singular at x.

Exterior Vectors

Definition

The **exterior (or wedge) product**, denoted by $\wedge$, is a multilinear operation on a vector space V that is associative and collapses $v \wedge v$ to 0.

Exterior products are anticommutative: $v \wedge w = -w \wedge v$.

For any vector space V over a field F, the set

$$\underbrace{V \wedge V \wedge \cdots \wedge V}_{n \text{ times}} = \bigwedge^{n} V$$

is also a vector space over F.

Associated Vectors

Definition

The **associated exterior vector** α_i to a hinge v_i is the exterior 2-vector $\alpha_i = (e_4 + p_i) \wedge v_i$.

It has been shown [Borcea, et. al] that a configuration is singular if and only if its associated exterior vectors don't span $\mathbb{R}^4 \wedge \mathbb{R}^4$.

Corollary

Singularity is preserved by linear transformations.

Theorem (Projective Intersection Theorem)

If there exists a line that intersects or is parallel to each hinge, the configuration is singular.

Helical Vector Fields

If a hinge structure is singular, the exterior vectors corresponding to hinges all lie in some hyperplane.

Theorem (Helicity Theorem)

A closed hinge structure is singular if and only if there is some vector field V on $\mathbb{R}^3$ of the form

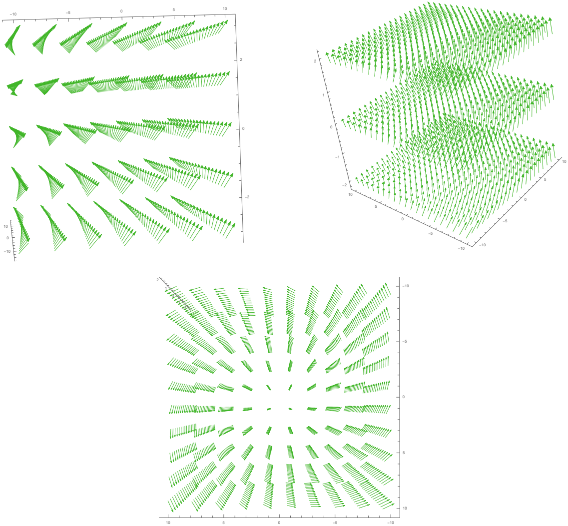
$$V(\vec{x}) = (\vec{x} - \vec{a}) \times \vec{n} + c * \vec{n}$$

that for every vertex p_i and hinge vector $v_i = p_{i+1} - p_i$ satisfies

$$V(p_i) \cdot v_i = 0$$

We call V a helical vector field, or helicity.

A Single Helicity



Reformulation

A helical vector field V has the property that if a straight line γ satisfies

$$\dot{\gamma}(t) \cdot V(\gamma(t)) = 0.$$

for some $t \in \mathbb{R}$, then it satisfies this equation for all real t. This is a very nice property.

Theorem (Helicity Theorem, Reformulated)

A closed hinge structure H is singular if and only if there exists a helical vector field to which H, considered as a piecewise linear loop, travels at all times perpendicular.

Hyperboloids

Take the set of all vectors based in some plane perpendicular to the symmetry axis. The span of all the vectors is a partition of $\mathbb{R}^3$ by lines. This is a very special property of V.

If we consider only the subset of vectors whose basepoints lie some fixed distance from the symmetry axis, we get a hyperboloid. This means that given our planar subset of $\mathbb{R}^3$, V induces a partition of $\mathbb{R}^3$ determined by this choice of plane.

Ruled Lines

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What Is the Hopf Fibration?

Definition

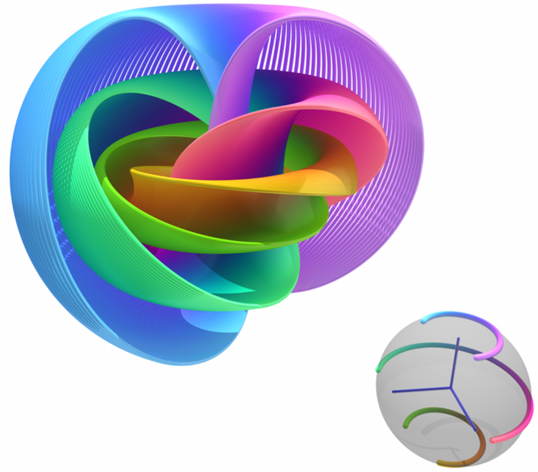
The **Hopf fibration** is a many-to-one continuous map from the three-sphere to the two-sphere where the preimage of each point is a great circle. It is a way to locally write S^3 as $S^1 \times S^2$.

If we embed $\mathbb{R}^3$ into $\mathbb{R}^4$ like before, we can "map" $\mathbb{R}^3$ onto S^3 by sending the point p to $\pm p/||p||$ and projectively completing.

Theorem (Hopf Fibration Theorem)

For $c = 1$ and the symmetry axis of V coinciding with the z-axis in $\mathbb{R}^3$, the ruled lines of the hyperboloids are the fibers of the Hopf fibration.

The Hopf Fibration



Classification by Fibrations

Definition

The **hinge-normal vector** n_i corresponding to the ith vertex p_i is given by $n_i = v_{i-1} \times v_i$.

Suppose that we translate all hinge-normal vectors along the symmetry axis so that their basepoints are coplanar. Then the configuration is singular if and only if every resulting vector lies along a ruled line.

Theorem (Fibration Theorem)

If we have a singular configuration and we translate our hinge-normal vectors like above, then there exists a fibration of S^3 over S^2 with great circles as fibers that sends the hinge-normal vectors to single points.

What Is a Javelin?

Definition

A **javelin** J_i corresponding to the ith vertex p_i is the line given by the intersection of the two planes determined the sets $\{p_{i-2}, p_{i-1}, p_i\}$ and $\{p_i, p_{i+1}, p_{i+2}\}$, where we take the indices modulo k if necessary.

Why Javelins?

- Question: How do you determine when a hinge structure is singular?
- Can move along javelins and stay singular.
- Know the value of the helical vector field on the javelins, if it exists

Algorithm for Determining Singularity

- Want to first find upward direction for helicity
- Value of the vector field on “sphere at infinity”
- Once upward direction is known, can project onto plane perpendicular to the upward direction
- Simple trigonometry to get the axis of the helicity
- Upward rise of vector field relative to plane perpendicular gives c

5 Points Theorem

- Take any five points in $\mathbb{R}^3$ that don't all lie in a plane
- Then the vectors $\hat{J}_i \times (p_{i+1} - p_{i-1})$ all lie in some plane

Space of Singular Configurations

- We can move vertices along javelins and stay singular
- Through such motions, we can collapse any hinge structure to a point
- Space of singular configurations is maybe a "cone"

Further Directions

- Flowing a hinge structure along javelins
- Configuration spaces of hinge structures
- Schubert calculus

Introduction

Vector Fields

Fibrations

Javelins

References

Borcea, C. and I. Streinu. *Singularities of Hinge Structures.* arXiv:0812.1373v1.