Summer@ICERM 2016 Projects

Focus of the program: The sample problems listed below center around modelling, analysis, and computational aspects of pattern formation in spatially extended systems. The projects involve tools from dynamical systems, numerical methods (including rigorous computations), and probability. Most of the references listed below can be accessed from `http://www.dam.brown.edu/people/sandsted/publications.php`.

1 Agent-based modelling of pattern-forming processes

Stripe formation in zebrafish: During their early development, zebrafish develop stripes formed of several types of colored pigment cells. Recent modelling efforts using agent-based models for the differentiation, movement, and death of a discrete population of pigment cells on the fish body have led to improved understanding of the mechanisms behind stripe formation in wild-type and mutant zebrafish. Stripes also form on the fins of zebrafish: the mechanism seems to differ from that on the main body and appears to involve movement of certain pigment cells along the bones of the fins. This project will focus on modelling pattern formation in fins using a combination of deterministic and stochastic equations for cells, parameter estimation from experiments, and numerical simulations of the resulting models. Reference: [13]

Localization of mRNA in frog oocytes: Establishing spatial directionality is an important part of the early development of all organisms. In Xenopus, directionality is achieved by localization of mRNA, first at the nucleus and then at the vegetal cortex. The process of moving mRNA cargo from the nucleus to the vegetal cortex involves several kinds of molecular motors, but their precise role and the role of anchoring of mRNA to the cortex is not well understood. This project would extend recent agent-based one-dimensional models to planar models that involve transport along complex networks of microtubules. Reference: [9]

Stationary distributions for chains of agents with birth and death: Agent-based models of cells typically involve (1) deterministic differential equations that govern their interactions and movement and (2) a stochastic component that governs cell birth and death. In particular, the number of cells varies stochastically due to birth and death processes. An interesting question is then to analyse when, and in what sense, the system reaches a stable equilibrium (which could be given by a stationary distribution of the underlying stochastic process). Using a combination of analytical and numerical techniques, this project will investigate a one-dimensional model for the interaction of particles that repel each other and analyse under what conditions on the repulsive force and the rate of birth/death a stationary distribution exists.

Propagation of lead in mammals: There exists a well-known linear 3-compartment ODE model ODE which describes the propagation of lead in mammals after ingestion. The 3 compartments are blood, tissue, and bone. The model implicitly assumes that the transfer rates between compartments are fixed; in particular, they do not depend upon the amount of lead in a given compartment. Using experimentally determined values for the transfer rates, and assuming that the ingestion rate is constant, the model predicts that after ingestion most of the lead in the body will be in the bones. The goal of the project will be to extend the model in several different directions, and to see if, and how, the various extensions effect this prediction. For example, what happens if

- the ingestion rates between compartments depends upon the amount of lead in a given compartment?
- a compartment, e.g., tissue, is assumed to have several sub-compartments, e.g., muscle, neural tissue, digestive, etc., each with its own transfer rate?

- the ingestion rate is not constant; in particular, suppose it is stochastic in some sense?
- the transfer rates depend upon age?

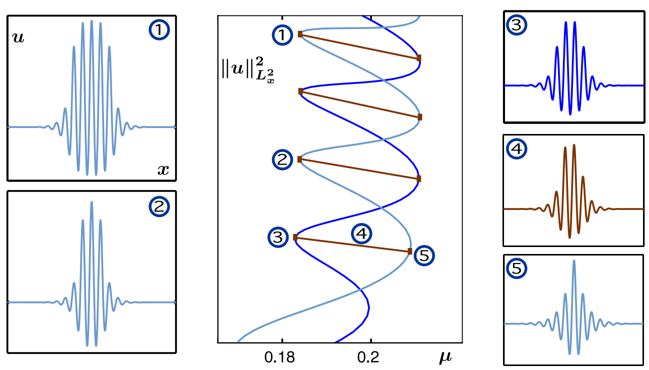
The goal of this project is to answer as these questions - and, perhaps more! - using a combination of analysis and numerical studies. References are: [5, 11]

2 Snaking in the Swift–Hohenberg equation

The Swift–Hohenberg equation

$$u_t = -(1+\Delta)^2 u - \mu u + \nu u^2 - u^3, \qquad x \in \mathbb{R}^d, \quad d = 1, 2$$

is a paradigm of pattern-forming systems. It exhibits planar roll or stripe patterns, domain-filling hexagon patterns, and a variety of localized patterns that resemble localized patches of rolls or hexagons. For the one-dimensional Swift–Hohenberg equation, it is known that stable localized roll patterns of different extent can coexist at the same parameter values: all these structures lie on the same bifurcation curves that "snake" back and forth in parameter space as the roll plateaus increase in length (see figure below) [1, 2, 4]. In addition, fully localized patterns, so-called spots, exist for the planar Swift–Hohenberg equation.



From localized to target patters: Numerical experiments show that localized spots can turn into nonlocalized "target" patterns as the parameter μ crosses through zero. It would be interesting to use geometric blow-up techniques complemented by rigorous numerical computations to prove that this novel transition occurs in the radial Swift–Hohenberg equation. Reference: [8]

From dimension 1 to $1+\epsilon$: It is known that the bifurcation diagrams of the one-dimensional and the planar Swift–Hohenberg equation are very different: the 1D equation exhibits snaking, while the bifurcation diagram for the 2d equation consists of infinitely many disconnected isolas. Numerical simulation indicate that the transition from connected to disconnected curves occurs when one goes from dimension $n=1$ to $n = 1+\epsilon$ for the radial equation

$$0 = -\left(1 + \frac{n-1}{r}\partial_r + \partial_r^2\right)^2 u - \mu u + \nu u^2 - u^3, \qquad r \geq 0. \tag{1}$$

The reason seems to be that drift along the cylinder of periodic orbits that exists for $n=1$ is induced as soon as $n>1$. The project would involve analysing this behavior using a combination of analytical dynamical-systems techniques for the vector field on the cylinder complemented by numerical simulations. Reference: [7]

Snaking for non-orientable Floquet bundles: The mathematical theories that were developed to explain the snaking diagram in the figure above all assume that the Floquet bundles along the roll patterns are topologically trivial. It is not clear how the bifurcation diagrams would change for non-orientable Floquet bundles. It would be interesting to analyse this case using, as in the orientable case, dynamical-systems techniques to solve equation (1). Reference: [2]

Emergence of localized defects inside roll patterns: The patterns discussed so far resemble localized patches of one-dimensional and planar roll or strip patterns. It is also possible to generate localized defects inside a roll pattern. Numerical computations indicate that such structures can emerge at fold bifurcations of roll patterns. Analysing this bifurcation using spatial dynamics would be a very interesting project amenable to undergraduates. References: [3, 4]

3 Patterns in planar systems

Low-frequency forcing of planar spiral waves: Spiral waves play an important role in sustaining certain cardiac arrhythmias such as tachycardia. In damaged cardiac tissue, pacemaker waves can get stuck and evolve into spiral waves that excite cardiac muscle cells with a much higher temporal frequency, thus potentially leading to fibrillation. Recent experiments have shown that low-energy far-field pacing can, for appropriate frequencies, remove spiral waves from inhomogeneous damaged tissue. The precise theoretical underpinnings for this mechanism are not known: this project would involve using theoretical and numerical approaches to study simpler toy problems that could help shed light on why far-field pacing is successful. A related problem that is of interest to diagnose tachycardia is the converse: which types of external forcing can lead to the emergence of spiral waves (presumably near damaged tissue that could be identified in this fashion and then ablated in surgery). Reference: [6]

Bifurcations of sources at Turing–Hopf bifurcations: Turing–Hopf bifurcations arise when a spatially homogeneous rest state becomes, at the same time, unstable to perturbations that are spatially periodic (but stationary in time) or time-periodic (but homogeneous in space). This opens up the possibility of creating patterns that are time-periodic and have a spatially periodic plateau in the center: such patterns have been observed in a number of experiments, including in Belousov–Zhabotinsky reactions. The goal of this project is to find these patterns analytically and to carry out a numerical study of their existence and stability properties. References: [10, 12]

References

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