Summer@ICERM 2017: Topological Data Analysis
Mini-course on Computational Topology

Lecturer: Henry Adams

Description: This course is an introduction to computational topology, and we motivate the course topics with recent applications of topology. The first application is to data analysis: the shape of a dataset often reflects important patterns within. Two such datasets with interesting shapes are a space of 3×3 pixel patches from optical images, which can be well-modeled by a Klein bottle, and the conformation space of the cyclo-octane molecule, which is a Klein bottle glued to a 2-sphere along two circles of singularities. We introduce homology as a way to measure the number of holes in a topological space, and persistent homology as a way to measure the number of holes in a dataset.

As a second application of topology, we will describe coverage problems in sensor networks. Suppose ball-shaped sensors wander in a bounded domain. A sensor doesn't know its location but does know when it overlaps a nearby sensor. We say that an evasion path exists in this sensor network if a moving intruder can avoid detection. Using zigzag persistent homology, one can compute a necessary condition for an evasion path to exist. However, no method with time-varying connectivity data (i.e. Čech complexes) as input can give necessary and sufficient conditions for the existence of an evasion path. In the setting of planar sensors that also measure weak rotation information, we discuss necessary and sufficient conditions for the existence of an evasion path, and pose an open question concerning Čech and alpha complexes.

Course topics include simplicial complexes, Čech and nerve complexes, Vietoris-Rips complexes, persistent homology, and zigzag persistent homology.

Format: The course consists of four blocks of 3 hours each. Each class will be split into a lecture of at most 60 minutes, followed by a short break. The rest of the time will be devoted to computing examples in the Javaplex software package.

Javaplex software package: `http://appliedtopology.github.io/javaplex/`

Javaplex tutorial: `http://www.math.colostate.edu/~adams/research/javaplex_tutorial.pdf`

Lecture 1: Applied topology & the cyclo-octane dataset.

We overview computational topology as applied to data analysis. One dataset is the conformation space of the cyclo-octane molecule, which is a Klein bottle glued inside a sphere along two circles of singularity.

Slides: AnIntroductionToAppliedAndComputationalTopology.pdf

Javaplex Exercises 1: Simplicial complexes and homology.

We build simplicial complexes, such as a torus, Klein bottle, or projective plane, as in Section 2 of the Javaplex tutorial.

Slides: Slides_HomologyIntroduction.pdf

Lecture 2: Optical image patches

In this lecture we consider 3×3 pixel patches from optical images. This dataset is well-modeled by a circle at a global choice of density estimate, by a three-circle shape at an intermediate choice of density estimate, and by a Klein bottle at a local choice of density estimate.

Javaplex Exercises 2: Filtered simplicial complexes and persistent homology.

We build filtered simplicial complexes and compute persistent homology, as in Section 3 of the Javaplex tutorial.

Lecture 3: Čech, Vietoris–Rips, and witness simplicial complexes.

In this class we study simplicial complexes in more detail, including Čech complexes, alpha complexes, Vietoris–Rips complexes, and witness complexes. We describe the Nerve Lemma for coverings of a topological space, Hausmann's theorem, Latschev's theorem, and the stability theorem for persistent homology.

Notes: Notes_Cech_VietorisRips.pdf

Javaplex Exercises 3: Metric spaces and geometric simplicial complexes.

We can encode a finite metric space in a computer either as a matrix of Euclidean points, or as a distance matrix. We produce filtered simplicial complexes on top of finite metric spaces; see Sections 4 and 5 of the Javaplex tutorial.

Lecture 4: Evasion paths in mobile sensor networks.

We introduce applications of topology to coverage problems in sensor networks and mobile sensor networks.

Slides: EvasionProblemSlides.pdf

Javaplex Exercises 4: Real data examples and coding tips. Examples with real are computed using Section 6 of the Javaplex software package. These include a dataset of range image patches, optical image patches, and the conformation space of the cyclo-octane molecule. Coding tips, including bottleneck distances and dense core subsets, will be shared from Section 7 and the appendices of the Javaplex tutorial.

Reading materials

The main reading reference will be the Javaplex tutorial (`http://www.math.colostate.edu/~adams/research/javaplex_tutorial.pdf`), though I will point students to more detailed references as dictated by their interests. General references are included below.

References

[1] Henry Adams and Gunnar Carlsson. Evasion paths in mobile sensor networks. *The International Journal of Robotics Research*, 34.1 (2015): 90–104.

[2] Gunnar Carlsson. Topology and data. *Bulletin of the American Mathematical Society*, 46(2): 255–308, 2009.

[3] Gunnar Carlsson and Vin De Silva. Zigzag persistence. *Foundations of computational mathematics*, 10(4): 367–405, 2010.

[4] Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian. On the local behavior of spaces of natural images. International journal of computer vision 76 (2008): 1–12.

[5] Herbert Edelsbrunner and John Harer. *Computational topology: An introduction.* American Mathematical Society, 2010.

[6] Shawn Martin, Aidan Thompson, Evangelos A. Coutsias, and Jean-Paul Watson. Topology of cyclo-octane energy landscape. *The journal of chemical physics* 132.23 (2010): 234115.