• SUMMER@ICERM 2021: BILLIARDS PROJECTS

Background

Dynamics is the study of systems in motion. A prototypical example of a dynamical system is a billiard ball as it bounces around a billiard table. To be even more concrete, consider a polygon in the plane with all of its angles rational (when measured in degrees) - such a polygon is called a rational polygon. Let's play billiards in this polygon, i.e. let's consider a ball (thought of as a single point) that moves in straight lines in the polygon until it hits an edge, at which point the ball deflects with incident angle equal to the reflected angle (i.e. the way that physics models a bouncing ball). We imagine that the vertices of our polygon contain "pockets" and so if the ball hits a vertex it "sinks into the pocket" and stops moving. This system is called a rational billiard table.

One dynamical question - called the illumination problem - that we could ask about a rational billiard table is whether there are two points in the polygon that do not illuminate each other, i.e. so that no billiard ball shot from the first point passes through the second point. A slightly harder version of this problem - the finite blocking problem - poses essentially the same question, but asks whether one can always shoot a billiard ball from the first point to the second point while avoiding an arbitrary finite collection of points.

As it turns out, the magic wand theorem of Eskin, Mirzakhani, and Mohammadi [**EMM15**] from Teichmüller dynamics (the definition of this mathematical discipline is momentarily deferred) provides a powerful tool for studying the illumination problem. Using the magic wand theorem, Leliévre, Monteil, and Weiss [**LMW16**] showed that given any fixed starting point on a billiard table, a talented enough billiard player could shoot the ball so that it traveled to every other point on the table with finitely many exceptions. But of course one may wonder whether there is a good reason for these finitely many exceptional points to exist.

To answer this question, consider the following approach to studying rational billiards. Suppose that a billiard ball strikes an edge of our polygonal billiard table. When this happens reflect the billiard table across the struck edge - forming the original polygonal table and its mirror image joined along an edge. If we follow the path of the billiard ball in the mirror image, it seems as if the ball travels along a straight line from its original position in the original polygonal table along a straight line through the edge and into the mirrored table. In other words, if we permit ourselves the use of mirrored copies of our table - billiard trajectories become straight lines. In Katok-Zemlyakov [**ZK75**], it is shown that by iterating this reflection procedure one can produce a translation surface on a closed surface - that is, a flat metric on a surface with finitely many conical singular points.

Every translation surface can be presented as a disjoint union of polygons in the plane with edges glued together by translation. Amazingly, there is a one-to-one correspondence between translation surfaces of genus g (i.e. ways to glue polygons together to make a surface with g handles) and $\Omega\mathcal{M}_g$ - the bundle of holomorphic one-forms over the moduli space of genus g Riemann surfaces. In particular, gluing together Euclidean polygons to make surfaces not only provides a lens through which to examine the dynamics of rational billiards, but gives a fruitful and intuitive way of studying complicated moduli spaces that lie at the crossroads of low-dimensional topology, algebraic geometry, and physics.

As remarked above, the unfolding of a rational billiard table is a translation surface and there is a correspondence between straight lines on the translation surface and billiard trajectories on the billiard table. This allows for a direct translation of the illumination problem and the finite blocking problem into the world of translation surfaces - for instance, the finite blocking problem becomes - when can two points on a translation surface be connected by a straight line? Perhaps it is less surprising now that Teichmüller dynamics - the branch of mathematics concerned with dynamics of Teichmüller geodesic flow on $\Omega\mathcal{M}_g$ - connects to rational billiards.

In fact the reverse is true as well - unexpected phenomena in Teichmüller dynamics (for instance the phenomena discovered in Eskin-McMullen-Mukamel-Wright [**EMMW**]) - was discovered by performing computer experiments with rational billiards!

We remark that by work of Paul Apisa and Alex Wright [**AW17**] that there is in fact a geometric reason why two points fail to illuminate each other on a rational billiard table; in particular, the geometry of the unfolding of a billiard table explains much about the dynamics of the underlying billiard table. It is precisely this circle of ideas around which we propose to base the program.

Proposed projects

Each of the following project proposals were chosen because they possess these three qualities -

1. The problems are simple to state.
2. The solution to any of the problems would represent progress in the study of either rational billiards, Teichmüller dynamics, or both.
3. Each problem is amenable to computer experimentation - specifically with packages in SAGE adapted to studying translation surfaces.

Project 1: The Finite Blocking Problem

Two points on a translation surface are said to be finitely blocked if there is a finite collection of points B so that any straight line that passes from one point to the other passes through B. By Apisa-Wright [**AW17**] there are only two kinds of surfaces that have (potentially) infinitely many pairs of finitely blocked points - squared-tiled surfaces and half-translation covers. The simplest example of a half-translation cover is a square-tiled surface - i.e. a flat surface that is formed by taking a disjoint collection of squares in the plane and identifying sides by translation to form a closed surface (the reason why we separate these two cases will be clear momentarily). In general, a half-translation cover admits a similarly simple geometric definition (in particular a definition that does not appeal to covering space theory), but we omit this definition here.

Problem 1. *Determine when a half-translation cover admits infinitely many pairs of finitely blocked points.*

For square-tiled surface any two points are finitely blocked from each other. So we pose a variant of the previous question.

Problem 2. *Determine when a square-tiled surface admits two points that cannot be joined by a straight line.*

The finite blocking problem was first considered from the Teichmüller dynamics perspective by Lelièvre, Monteil, and Weiss [**LMW16**] and the illumination problem has been more recently considered by Wolecki [**Wol19**]. We propose having the students conduct computer experiments on square-tiled surfaces and half-translation covers in SAGE (where there is already an algorithm for generating and studying square-tiled surfaces), formulating a conjecture, and then potentially imitating some of the ideas in Apisa [**Api**], Apisa-Wright [**AW17**], or Apisa [**Api17**] to verify their conjecture.

Project 2: Square-Tiled Surfaces

A cylinder on a translation surface is a Euclidean cylinder that is embedded into the surface with respect to the flat metric. Cylinders have been one of the principal ways to study Teichmüller dynamics in recent years (see for instance Wright [**Wri15**]) and consequently there are many tools that have already been developed in SAGE to study cylinders, especially cylinders on square-tiled surfaces.

The following problem arose in Forni (though in a different guise, see [**For02**] and [**For06**]), who developed a criterion to connect the number of disjoint cylinders on a translation surface to the Lyapunov spectrum of Teichmüller geodesic flow.

Problem 3. *A square-tiled surface is called geminal if every cylinder direction consists of at most two cylinders. Every square-tiled surface is specified by two permutations - one that, after labelling each square - indicates how to glue the top of one square to the bottom of the next and another that indicates how to glue the right edge of one square to the left edge of another. Computational evidence suggests that if a square-tiled surface is geminal then the subgroup generated by these two permutations contains an index four Abelian subgroup. Prove or disprove.*

Aside from computer experimentation, recent related work of Matheus-Yoccoz [**MY10**] may provide some useful techniques for studying this problem.

The following question was suggested by Vincent Delecroix and is related to topological recursion.

Problem 4. *Let a matrix in* $\mathrm{SL}(2,\mathbb{Z})$ *act on a square-tiled surface by acting on the disjoint collection of squares in the plane that comprise the surface. It* *turns out that this action takes square-tiled surfaces to square-tiled surfaces. What invariants distinguish these* $\mathrm{SL}(2,\mathbb{Z})$ *orbits?*

The following question is important in understanding the geometry of strata and was considered by Lelievre and Weiss [**LW15**].

Problem 5. *When can a square tiled surface be presented as a convex polygon with opposite sides identified?*

Project 3: Periodic Points on Veech Surfaces

The group $\mathrm{GL}(2,\mathbb{R})$ acts on $\mathbb{R}^2$ by matrices. Since a translation surface is composed of a disjoint union of polygons in $\mathbb{R}^2$ with sides identified by translation, the $\mathrm{GL}(2,\mathbb{R})$ action on the plane induces a $\mathrm{GL}(2,\mathbb{R})$ action on translation surfaces. A translation surface is called a Veech surface if the Veech group - i.e. the subgroup of matrices in $\mathrm{SL}(2,\mathbb{R})$ that take the translation surface to itself -forms a lattice in $\mathrm{SL}(2,\mathbb{R})$. A point on a Veech surface is called a periodic point if its orbit under the Veech group is finite. By Eskin-Filip-Wright ([**EFW18**]), the only Veech surfaces that have infinitely many periodic points are square-tiled surfaces. Computing the collection of periodic points is connected to determining the holomorphic sections of the universal curve restricted to an algebraic curve (called the Teichmüller curve associated to a Veech surface) in the moduli space of Riemann surfaces (see Apisa [**Api**]).

Problem 6. *The following are four infinite families of Veech surfaces (the first two of which are encoded in SAGE as a collection of polygons with sides identified by translation). Determine the collection of periodic points on each surface in the family:*

1. *The Bouw-Möller curves (see Bouw-Möller* [**BM10**] *and Hooper* [**Hoo13**]*).*
2. *The Prym eigenforms (see McMullen* [**McM06**]*).*
3. *The Gothic eigenforms (see McMullen-Mukamel-Wright* [**MMW17**]*).*
4. *The unfoldings of rational triangles (see Mirzakhani-Wright* [**MW18**]*).*

For a given surface the collection of techniques in Apisa [**Api**] - which could even be implemented as code in SAGE - should produce a finite set of points on the translation surface that contain the set of periodic points. Computer experiments would then be used to determine which elements of this set were actually periodic points and to facilitate formulating conjectures about which Veech surfaces in the family have periodic points.

Project 4: Periodic Billiard Paths

When does a polygon contain a periodic billiard path? In other words, when can a billiard ball be shot around a billiard table and return to its original position? Masur showed that every rational polygon possesses this property. However it is unknown if every triangle possesses this property! Schwartz showed that if a triangular billiard table has its maximal angle less than 100 degrees, then it possesses a periodic billiard path.

Problem 7. *Do all triangles possess a periodic billiard trajectory?*

Problem 8. *Computer experiments suggest that every rational quadrilateral possesses a periodic billiard trajectory that begins by hitting one edge of the quadrilateral orthogonally. Prove or disprove.*

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