Computational Polygonal Billiards — Summer@ICERM 2021

Final Presentation Schedule

Hecke Eigenforms to Flat Atlases

Zawad Chowdhury[1], ***Elaine Danielson***[2], ***Aidan Mager***[3], ***Hamilton Wan***[4]

[1] Massachusetts Institute of Technology, [2] University of Florida, [3] University of Washington, [4] Yale University

There is a correspondence between Riemann surfaces equipped with a holomorphic 1-form and flat surfaces (polygons in the plane with sides identified by translation). Since weight two Hecke eigenforms associated to $\Gamma_0(N)$ correspond to Riemann surfaces equipped with a holomorphic 1-form, it is an interesting problem to compute their polygonal representation. The problem is motivated by finding novel $\mathrm{GL}(2, \mathbb{R})$ orbit closures in strata of translation surfaces. We discuss our work towards developing an algorithm that inputs a Hecke eigenform (given as a finite initial string of a q-expansion) and outputs the polygonal representation.

Closed Geodesics on Dilation Surfaces

Catherine Cui[1], ***Victor Ginsburg***[2], ***Veronica Kirgios***[3], ***Vanessa Lin***[4]

[1] Harvard University, [2] Pennsylvania State University, [3] University of Notre Dame, [4] University of North Carolina

The geometry of translation surfaces is an active research area in Teichmüller dynamics. Geodesics on translation surfaces have been well-studied, but much less is known about the geodesics on a dilation surface, a generalization of a translation surface. We demonstrate that the closed geodesics on a dilation surface can have dramatically different behavior compared to closed geodesics on a translation surface.

Hyperbolic Staircases: Periodic paths on $(2n+1)$-gons

Mei Rose Connor[1], ***Michael Kielstra***[2], ***Zachary Steinberg***[3], ***Chenyang Sun***[4]

[1] Stony Brook University, [2] Harvard University, [3] Amherst College, [4] Williams College

Dynamics on billiards have continued to provide an active area of research for over a century. In recent papers, Davis-Fuchs-Tabachnikov and Davis-Lelièvre have independently developed methods to classify the periodic trajectories on the pentagon by identifying slopes of periodic directions with points in the Poincaré disk, either by Möbius transformations of the pentagon in $\mathbb{H}$, and by viewing the double pentagon as a translation surface called the 'golden L'. We connect and unify these two approaches, and use our unification of these results to generalize them to arbitrary 2n+1 sided regular polygons.

Periodic Orbits of Affine Interval Exchange Transformations

Kelly Chen[1], ***Zachary Steinberg***[2], ***Cameron Thomas***[3]

[1] Massachusetts Institute of Technology, [2] Amherst College, [3] Morehouse College

Dilation surfaces are a relatively new class of surfaces which can be represented as collections of polygons in the plane with edges identified by translation and dilation. Although they are a generalization of translation surfaces, they exhibit vastly different dynamics, for example trajectories that attract onto periodic orbits. Trajectories on dilation surfaces are closely related to affine interval exchange transformations (AIETs); in particular, periodic trajectories correspond to periodic orbits of AIETs. We show that almost every AIET has finitely many periodic orbits, and we conjecture based on computational and theoretical evidence that almost every AIET has at least one periodic orbit.

Language Complexity of Billiards

Jessica Bennett[1], ***Catherine Cui***[2], ***Elaine Danielson***[2], ***Veronica Kirgios***[4]

[1] Brown University, [2] Harvard University, [3] University of Florida, [4] University of Notre Dame

One way to measure the "complexity" of a billiard table is to enumerate all sequences of sides that a billiard trajectory could hit. Previous studies of language complexity focused on tables that tile the Euclidean plane, such as the equilateral triangle. In contrast, we will describe an approach for studying the language complexity of tables that do not tile the Euclidean plane, like the regular pentagon. We will describe our program that associates an infinite-order ideal tiling of the hyperbolic plane to each regular n-gon, providing a geometric interpretation of achievable sequences of bounces on the original tables.

The J-invariant as a tool for detecting Veech Surfaces and the Combinatorics of Lattice Hexagons

Mei Rose Connor[1], ***Brin Harper***[2], ***Hamilton Wan***[3], ***Hanna Yang***[2]

[1] Stony Brook University, [2] Massachusetts Institute of Technology, [3] Yale University

Veech surfaces are translation surfaces with closed $\mathrm{GL}(2, \mathbb{R})$ orbits and optimal dynamics. The classification of Veech surfaces is an open question even in low genus. We study the power of the Kenyon-Smillie J-invariant (a 2-dimensional analogue of the Dehn invariant used to solve Hilbert's third problem) as a tool for distinguishing between Veech and non-Veech surfaces. Making use of the J-invariant, we approach the classification problem in the stratum $\mathcal{H}(4)$ through the lenses of convexly presented 12-gons and unfoldings of quadrilaterals. We also look at square tiled surfaces, which are always Veech, and present a count of such surfaces admitting convex presentations.

Long and Short Trajectories in the Double Pentagon

Sam Everett[1], ***Vanessa Lin***[2], ***Aidan Mager***[3]

[1] University of Colorado, Boulder, [2] University of North Carolina, [3] University of Washington

In any periodic direction, the double pentagon translation surface decomposes into two cylinders, one longer than the other. For each such direction and a specified starting point p, McMullen asked whether one could determine whether the periodic trajectory through p belongs to the long or short cylinder. In this talk we will present an algorithm resolving this question.

Blocking and Periodic Points on Veech Surfaces

Jessica Bennett[1], ***Zawad Chowdhury***[2], ***Sam Everett***[3], ***Destine Lee***[4]

[1] Brown University, [2] Massachusetts Institute of Technology, [3] University of Colorado, Boulder, [4] Columbia University

There are many questions one can ask about the dynamics of points on translation surfaces, such as "when are two points blocked from each other under straight-line flow," or "which points are periodic under the action of the Veech group." This summer we investigated two specific instances of these problems: finite blocking on cyclic covers of the regular octagon, and periodic points on Prym eigenforms in genus three. In this talk, we first present an algorithm, suitable for a broad class of surfaces including many Prym eigenforms, that produces a finite list of periodic points. We then answer the finite blocking problem in the case of double covers of the octagon.

Towards the Asymptotic Language Complexity of the Regular Hexagon

***Michael Kielstra*[1], *Chenyang Sun*[2], *Cameron Thomas*[3]**

[1] Harvard University, [2] Williams College, [3] Morehouse College

The complexity of a language is a function of n which gives the number of words of length n in the language. In the study of trajectories in polygonal billiard tables, it is common to analyze the complexity of billiard trajectories by viewing them as words expressed in the labels of the sides of the table, and analyzing the complexity of the language formed in the process. We present work towards a proof of a conjecture that the language complexity of the hexagon is, asymptotically, $\frac{621}{32\pi^2}n^3$. This involves calculating the asymptotic density of lattice points on the hexagonal grid and making progress towards finding the number of trajectories on the grid bounded by some given length.