Tripod Configurations

Eric Chen, Nick Lourie, Nakul Luthra

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Definition 1
A tripod configuration of a closed plane curve $\gamma$ consists of three normal lines dropped from $\gamma$ meeting at a single point (the tripod center) and making angles of $\frac{2\pi}{3}$.

Definition 2
A locally convex curve is a curve with nowhere vanishing curvature.
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A locally convex curve is a curve with nowhere vanishing curvature.
Existence of Tripod Configurations (Regular Polygons)

Definition 3:
A tripod configuration of a closed polygon \( P \) consists of three lines dropped from vertices of \( P \) meeting at a single point and making angles of \( \frac{2\pi}{3} \) such that each line is normal to a support line of \( P \) at the vertex through which it passes.

Theorem 4 (Summer@ICERM 2013):
A regular polygon with \( n \) vertices has \( n \) tripod configurations if \( 3 \nmid n \) and \( \frac{n}{3} \) tripod configurations if \( 3 \mid n \).
Definition 3

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Theorem 4 (Summer@ICERM 2013)
A regular polygon with $n$ vertices has $n$ tripod configurations if $3 \nmid n$ and $\frac{n}{3}$ tripod configurations if $3 \mid n$. 
Existence of Tripod Configurations (Plane Curves)

Theorem 5 (Tabachnikov 1995)
For any smooth convex closed curve there exist at least two tripod configurations.

Theorem 6 (Kao and Wang 2012)
If $\gamma$ is a closed locally convex curve with winding number $n$, then $\gamma$ has at least $\frac{n}{2}$ tripod configurations.

Theorem 7 (Summer@ICERM 2013)
If $\gamma$ is a closed locally convex curve with winding number $n$, then $\gamma$ has at least $2\left\lfloor \frac{n}{2} \right\rfloor + 3$ tripod configurations.

Every plane curve has a tripod configuration.

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1. If $\gamma$ is a closed locally convex curve with winding number $n$, then $\gamma$ has at least $2\left\lfloor \frac{n^2+2}{3} \right\rfloor$ tripod configurations.
2. Every plane curve has a tripod configuration.
Given a triangle $ABC$, the largest equilateral triangle circumscribing it is its antipedal triangle with respect to its first isogonic center.

Lemma 9
Let $p$, $q$, $r$ be noncollinear points on a closed plane curve $γ$ and let $T$ be an equilateral triangle with each side passing through one of the three points. Then the equilateral triangle circumscribing $γ$ with sides parallel to $T$ is at least as large as $T$.

Lemma 10
The largest equilateral triangle circumscribing a closed plane curve meets the curve exactly once per side.
Lemma 8

Given a triangle $ABC$, the largest equilateral triangle circumscribing it is its antipedal triangle with respect to its first isogonic center.
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The largest equilateral triangle circumscribing a closed plane curve meets the curve exactly once per side.
At least how many critical points does a smooth function on a circle have?
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• At least how many critical points does a smooth function on $S^n$ have?
Let $f : T^2 \to \mathbb{R}$ be the height function, or projection onto the z-axis.
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let \( M^a = \{ x \in T^2 \mid f(x) < a \in \mathbb{R} \} \).

\( M^a \) is every point in \( T^2 \) below the height of \( a \).
The bottom is a critical point, a local minimum

The index of the Hessian at the local minimum is zero

$M^{a_1}$ is homotopic to a 0-cell
The bottom point of the inner circle is a critical point, a saddle point.
The index of the Hessian at the point is one.
\(M^{a_1}\) is homotopic to a disk with a 1-cell attached.
We are not at a critical point of the function

$M^{a_3}$ is homotopic to $M^{a_2}$, there has been no change in the homotopy class.
The top point of the inner circle is a critical point, a saddle point
The index of the Hessian at the point is one
$M^a$ is homotopic to a cylinder with a 1-cell attached
The top point of the torus is a critical point, a local maximum.
- The index of the Hessian at the point is two.
- \( M^{a_5} \) is homotopic to a punctured torus with a 2-cell attached.
Definition 11 (non-degenerate)

A critical point of a function $f$ is said to be non-degenerate if the Hessian of $f$ at $p$ is non-singular.
Basic Definitions

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A critical point of a function $f$ is said to be non-degenerate if the Hessian of $f$ at $p$ is non-singular.

Definition 12 (Morse index)
The Morse index of a critical point $p$ of a function $f$ is the index of the Hessian of $f$ at $p$, i.e. the dimension of the largest subspace on which the Hessian is negative definite.
**Lemma 13 (The Morse Lemma)**

*If p is a non-degenerate critical point of f, then \( \exists \phi \), a chart of M, such that \( x_i(p) = 0 \forall i \) and \( f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 \) where \( k \) is the index of p.*
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Theorem 14

Given $a < b$, if $f^{-1}[a, b]$ is compact and no critical values lie in the interval $[a, b]$ then $M^a$ is a deformation retract of $M^b$. 
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Theorem 14

Given $a < b$, if $f^{-1}[a, b]$ is compact and no critical values lie in the interval $[a, b]$ then $M^a$ is a deformation retract of $M^b$.

Most importantly, this implies that $M^a$ has the same homotopy class as $M^b$. 
The preceding lemma and theorem allow us to prove the following:

**Theorem 15**

If $p$ is a non-degenerate critical point with Morse index $k$, $f(p) = a$ and then if we choose $\epsilon$ small enough so that $f^{-1}[a - \epsilon, a + \epsilon]$ is compact and contains no critical values other than $p$, then $M^{a+\epsilon}$ is of the homotopy class of $M^{a-\epsilon}$ with a $k$-cell attached.
The preceding theorems show that the number of critical points is equal to the number of cells in the cell structure of the manifold defined by the function $f$. 
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This equality of the cell structure and the critical points can be used to prove certain inequalities about critical points from the homotopy class of a manifold.
Corollary 16

Let \( C^\lambda \) denote the number of critical points of \( f \) with Morse index \( \lambda \).

\[
\sum (-1)^\lambda C^\lambda = \chi(M)
\]
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Let $C^\lambda$ denote the number of critical points of $f$ with Morse index $\lambda$.

$$\sum (-1)^\lambda C^\lambda = \chi(M)$$

Corollary 17

$C^\lambda \geq b_\lambda(M)$

Where $b_\lambda(M)$ is the $\lambda$'th Betti number of $M$. 
Our work mainly uses results about Morse Theory on manifolds with boundary.

Francois Laudenbach published a paper on Morse Theory on manifolds with boundary in 2010.

He derives Morse inequalities using a chain complex defined on manifolds with boundary.
Two new types of critical points appear on the boundary, when the gradient of $f$ is normal to it.
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Type N (Neumann) critical points occur when the gradient of $f$ points inward along the boundary.

Type D (Dirichlet) critical points occur when the gradient of $f$ points outward along the boundary.
Two new types of critical points appear on the boundary, when the gradient of $f$ is normal to it.

- Type N (Neumann) critical points occur when the gradient of $f$ points inward along the boundary.
- Type D (Dirichlet) critical points occur when the gradient of $f$ points outward along the boundary.
Morse Theory on Manifolds with Boundary

We obtain new Morse inequalities from the following fact:

**Theorem 18**

Let $C^\lambda$ and $B^\lambda$ be the number of critical points of index $\lambda$ in the interior and on the boundary (of D type) respectively. If $P(t)$ is the Poincare polynomial of $M$, $C(t) = \sum C^\lambda t^\lambda$, and $B(t) = \sum B^\lambda t^{\lambda+1}$, then:

$$B(t) + C(t) - P(t) = (1 + t)Q(t)$$

Where $Q(t)$ is a polynomial with non-negative integer coefficients.
We show that any convex curve in the plane and any convex curves sufficiently close to a circle in spherical and hyperbolic geometry have at least two tripod configurations.
Morse Theory and Tripods - Planar Case

\[ f : S^1 \times S^1 \times S^1 \times \overline{D} \rightarrow \mathbb{R}, (x, y, z, p) \mapsto d(x, p) + d(y, p) + d(z, p) \]

Critical points of \( f \) where \( p \in \mathbb{D} \) are tripod points.
Parallel curves have the same tripod configurations. “Boundary” critical points of $f$ when $p \in \partial D$ and the evolute is small:
Morse Theory and Tripods - Planar Case

Use osculating circles to approximate the curve near “boundary” critical points of $f$ to compute the Hessian up to second order approximation:
\[ p = (-r \cos \alpha, -r \sin \alpha) \approx (-r(1 - \frac{\alpha^2}{2}), -r\alpha) \]

\[ x = ((r + \epsilon) \cos \beta, -(r + \epsilon) \sin \beta) \approx (-r + \epsilon)(1 - \frac{\beta^2}{2}), -(r + \epsilon)\beta) \]

\[ y = (d + R \cos \gamma, R \sin \gamma) \approx (d + R(1 - \frac{\gamma^2}{2}), R\gamma) \]

\[ z = (d + R \cos \delta, R \sin \delta) \approx (d + R(1 - \frac{\delta^2}{2}), R\delta). \]
Morse Theory and Tripods - Planar Case

\[
\begin{pmatrix}
\frac{r(r+\epsilon)}{\epsilon} - \frac{2r(d+R)}{d+R+r} & -\frac{r(r+\epsilon)}{\epsilon} & \frac{Rr}{d+R+r} & \frac{Rr}{d+R+r} \\
-\frac{r(r+\epsilon)}{\epsilon} & -\frac{2r(d+R)}{d+R+r} & 0 & 0 \\
\frac{R}{d+R+r} & 0 & -\frac{R(d+r)}{d+R+r} & 0 \\
\frac{R}{d+R+r} & 0 & 0 & -\frac{R(d+r)}{d+R+r}
\end{pmatrix}
\]

**Theorem 19**

Given an \( n \times n \) matrix \( A \), call the determinant of the \( i \times i \) upper-left corner the \( i \)th leading minor and denote it by \( d_i \). Assume that \( A \) is symmetric and the \( d_i \)'s are non-zero...Then \( d_1, d_2/d_1, d_3/d_2, \ldots, d_n/d_{n-1} \) are diagonal entries in a diagonalization of \( A \).
Morse Theory and Tripods - Planar Case

Morse indices
Case 1:

\[
\begin{cases}
4, & d > 0 \\
3, & d < 0
\end{cases}
\]

Case 2:

\[
\begin{cases}
3, & d > 0 \\
2, & d < 0
\end{cases}
\]
Theorem 18

Let $C^\lambda$ and $B^\lambda$ be the number of critical points of index $\lambda$ in the interior and on the boundary (of D type) respectively. If $P(t)$ is the Poincare polynomial of $M$, $C(t) = \sum C^\lambda t^\lambda$, and $B(t) = \sum B^\lambda t^{\lambda+1}$, then:

$$B(t) + C(t) - P(t) = (1 + t)Q(t)$$

Where $Q(t)$ is a polynomial with non-negative integer coefficients.

A convex curve has as many “short” diameters ($d < 0$) as “long” diameters ($d > 0$). Let $m$ be the number of “short” (“long”) diameters.

$$2mt^5 + 6mt^4 + 2mt^4 + 6mt^3 + C(t) - (1 + t)^3 t^2 = (1 + t)Q(t)$$

$$(1 + t)(2mt^4 + 6mt^3) + C(t) - (1 + t)^3 t^2 = (1 + t)Q(t)$$

$$6(1 + t)|C(t)$$

So the curve has at least 2 tripod configurations.
Morse Theory and Tripods - Spherical Case

The “small evolute” condition becomes a “close to a circle” condition on the sphere. Use a second order approximation with geodesic circles.

\[ \text{op} = r \]
\[ \text{ox} = r + \varepsilon \]
\[ \text{oq} = d \]
\[ \text{qy} = \text{qz} = R \]
Morse indices (same as in planar case)

Case 1:

\[
\begin{cases}
4, & d > 0 \\
3, & d < 0
\end{cases}
\]

Case 2:

\[
\begin{cases}
3, & d > 0 \\
2, & d < 0
\end{cases}
\]
Use a second order approximation with geodesic circles in the Poincaré disc model.
Morse Theory and Tripods - Hyperbolic Plane Case

Morse indices (same as in planar case)
Case 1:

\[
\begin{cases}
4, & d > 0 \\
3, & d < 0 
\end{cases}
\]

Case 2:

\[
\begin{cases}
3, & d > 0 \\
2, & d < 0 
\end{cases}
\]