FINITE FREE RESOLUTIONS AND ROOT SYSTEMS

JERZY WEYMAN

1. Introduction

These are short notes meant for the participants of the August 2020 ICERM workshop on Finite Free Resolutions and Representation Theory. They are meant to be the guide to a discussion on resolutions of length 3 and resolutions of Gorenstein ideals of codimension 4. I thank Claudia Miller for going through the preliminary version of the notes and suggesting many improvements.

2. Notation

Let us set some basic notation.

- R is a commutative ring (usually Noetherian)
- $\mathbb{F}_\bullet$ is a complex

$$\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \to \ldots \to F_i \xrightarrow{d_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{d_1} F_0$$

- F_i are free R-modules of rank f_i, $r_i := \text{rank } d_i$,
- $\underline{r} = (r_1, \ldots, r_n)$ is the **rank sequence** of $\mathbb{F}_\bullet$, where r_i is defined as the maximal size of the nonvanishing minors of d_i, and
- $\underline{f} = (f_0, \ldots, f_n)$ is the **format** of $\mathbb{F}_\bullet$.
- We know that rank $F_i = r_i + r_{i+1}$ $(1 \le i \le n)$ with $r_{n+1} = 0$,
- WLOG we can assume rank $F_0 = r_1$.

2.1 Definition. Let us fix a format $\underline{f} = (f_0, \ldots, f_n)$ and the corresponding rank sequence $\underline{r} = (r_1, \ldots, r_n)$. We consider the pairs $(R, \mathbb{F}_\bullet)$ with R Noetherian, $\mathbb{F}_\bullet$ free acyclic of the given format and rank sequence.

We say that the pair $(R_{gen}, \mathbb{F}_\bullet^{gen})$ (the ring R_{gen} not necessarily Noetherian) is **generic** for format $\underline{f}$ and rank sequence $\underline{r}$ if

(1) $\mathbb{F}_\bullet^{gen}$ is acyclic,
(2) for every pair $(S, \mathbb{G}_\bullet)$ of the same format and rank, there exists a (not necessarily unique) map $\phi : R_{gen} \to S$ such that $\mathbb{G}_\bullet = \mathbb{F}_\bullet^{gen} \otimes_{R_{gen}} S$.

2.2 Theorem. *For every format $\underline{f}$ and the corresponding rank sequence $\underline{r}$ there exists a (not necessarily unique) generic pair*

$$(R_{gen}, \mathbb{F}_\bullet^{gen}).$$

2.3 Remark. One would like to know R_{gen} explicitly and whether it is Noetherian.

We now recall some old results

2.4 Proposition.

(1) *The case $n = 1$ is trivial.*
(2) *The case $n = 2$ solved by Hochster (1975) (see [H]). The generic ring always exists and the homomorphism ϕ is even unique. To construct it, one just adds fractions (Buchsbaum-Eisenbud multipliers, see [BE1]) to a generic complex.*

Date: August 3, 2020.

3. Generic rings for resolutions of length 3

We specialize to the case $n = 3$, We have a resolution

$$0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

Starting from a generic complex we add new elements to the ring by killing new cycles that appear in homology. The lifting of a cycle in $H_1(\mathbb{F}_\bullet)$ is not unique because lifting of a map $u : X \to F_1$ to a map $v : X \to F_2$ can be modified by any lifting $w : X \to F_3$.

$$\begin{array}{ccccccc} F_3 & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 \\ & & & {}_v\nwarrow & \uparrow u & & \\ & & & & X & & \end{array}$$

This leads to a notion of **defect** of a lifting, and makes the process more complicated.

Without going into unnecessary details let me just say I did construct a specific generic ring $\hat{R}_{gen}$ which carries a bigger symmetry. Consider the triple $(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1)$. Consider the corresponding T-shaped graph $T_{p,q,r}$

$$\begin{array}{ccccccccccccc} x_{p-1} & - & x_{p-2} & \ldots & x_1 & - & u & - & y_1 & \ldots & y_{q-2} & - & y_{q-1} \\ & & & & & & | & & & & & & \\ & & & & & & z_1 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & z_2 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & \vdots & & & & & & \\ & & & & & & z_{r-2} & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & z_{r-1} & & & & & & \end{array}$$

There is a Kac-Moody Lie algebra $\underline{g}(T_{p,q,r})$ corresponding to this graph. Let us treat it as a black box, we will see examples soon.

We distinguish a simple root corresponding to vertex z_1, the green vertices are the root system of $SL(F_1)$, and the blue ones are the root system of $SL(F_3)$.

Distinguishing a simple root corresponding to z_1 defines a $\mathbf{Z}$-grading

$$\underline{g}(T_{p,q,r}) = \bigoplus_{i \in \mathbf{Z}} \underline{g}_i(T_{p,q,r}).$$

3.1 Theorem. *(see [W]). There exists a generic ring $\hat{R}_{gen}$ for the rank sequence (r_1, r_2, r_3) which deforms to a ring R_{spec} which has a multiplicity free action of the Lie algebra $\underline{gl}(F_2) \times \underline{gl}(F_2) \times \underline{g}(T_{p,q,r})$. The ring $\hat{R}_{gen}$ is Noetherian if and only if $T_{p,q,r}$ is a Dynkin graph, i.e., a graph of type A_n, D_n, E_6, E_7, E_8.*

3.2 Remark.

(1) There exists a flat deformation with the general fibre isomorphic to the ring $\hat{R}_{gen}$ with a special fibre isomorphic to a ring $\hat{R}_{spec}$ which has a multiplicity free action of Lie algebra $\underline{gl}(F_2) \times \underline{gl}(F_0) \times \underline{g}(T_{p,q,r})$

(2) This deformation is not really necessary and it is a technical point. I believe the ring $\hat{R}_{gen}$ carries a multiplicity free action of Lie algebra $\underline{gl}(F_2) \times \underline{gl}(F_2) \times \underline{g}(T_{p,q,r})$. I will use this fact below.

3.3 Theorem. *(see [W]) If we denote the (well known) decomposition of the Buchsbaum-Eisenbud multiplier ring R_a by*

$$R_a = \oplus_{\lambda\in\Lambda} S_{\alpha(\lambda)}F_3 \otimes S_{\beta(\lambda)}F_2 \otimes S_{\gamma(\lambda)}F_1 \otimes S_{\delta(\lambda)}F_0$$

where $S_\mu F_i$ denotes the Schur functor (tensored with some power of determinant, so negative indices appear) and Λ is a lattice (specified in [W], Proposition 6.8) then the ring $\hat{R}_{spec}$ has a companion decomposition

$$\hat{R}_{spec} = \oplus_{\lambda\in\Lambda} S_{\beta(\lambda)}F_2 \otimes S_{\delta(\lambda)}F_0 \otimes V_{\mu(\alpha(\lambda),\gamma(\lambda))}$$

where $V_{\mu(\alpha(\lambda),\gamma(\lambda)}$ is the lowest weight representation of $\underline{g}(T_{p,q,r})$. Here $\mu(\alpha(\lambda),\gamma(\lambda))$ is the weight whose labels corresponding to nodes at arms p,q (corresponding to F_1) are those of $\gamma(\lambda)$, those at the arm r (except for z_1) are those of $\alpha(\lambda)$ and the label at node z_1 ia also determined by $\alpha,\beta,\gamma,\delta$.

Now we specialize to the formats of resolutions of cyclic modules, i.e.,

$$\operatorname{rank} F_0 = r_1 = 1.$$

Equivalently, $p = 2$.

3.4 Remark. The Dynkin formats for resolutions of cyclic modules are:

(1) A_n: $(1, 3, n, n-2)$,
(2) D_n: $(1, n, n, 1)$ and $(1, 4, n, n-3)$,
(3) E_6: $(1, 5, 6, 2)$,
(4) E_7: $(1, 5, 7, 3)$ and $(1, 6, 7, 2)$,
(5) E_8: $(1, 5, 8, 4)$ and $(1, 7, 8, 2)$.

The extended Dynkin formats are:

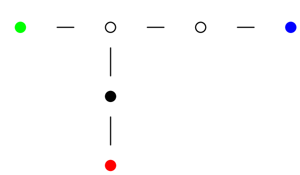
(1) $\hat{E}_7$: $(1, 6, 8, 3)$,
(2) $\hat{E}_8$: $(1, 5, 9, 5)$ and $(1, 8, 9, 2)$.

The Buchsbaum-Eisenbud multiplier ring is generated by entries of three matrices: d_3, (we think of it as a tensor from $F_3^* \otimes F_2$), d_2 (we think of it as a tensor from $F_2^* \otimes F_1$), and a_2 (we think of it as a tensor from $\bigwedge^{r_2} F_1$). Lying over those three in $\hat{R}_{gen}$ we have three critical representations

$$W(d_3) = F_2^* \otimes V(\omega_{z_{r-1}}),$$
$$W(d_2) = F_2 \otimes V(\omega_{y_{q-1}}),$$
$$W(a_2) = \mathbf{C} \otimes V(\omega_{x_{p-1}}).$$

Here $V(u)$ for a vertex u denotes the fundamental representation of $\underline{g}(T_{p,q,r})$ corresponding to that vertex.

In terms of Dynkin diagram (illustrated by graph E_6) these fundamental representations correspond to colored vertices

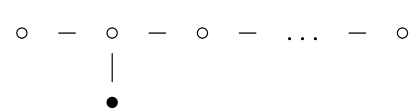


Let us look at some examples.

3.5 Proposition. *Consider the format $(1, n, n, 1)$. These are complexes*

$$0 \to R \to R^n \to R^n \to R.$$

The graph $T_{p,q,r}$ is the graph of type D_n.



The corresponding Lie algebra is

$$\underline{so}(F_1 \oplus F_1^*) = \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1$$

with

$$\underline{g}_{-1} = F_3 \otimes \bigwedge^2 F_1^*, \underline{g}_0 = \underline{gl}(F_1), \underline{g}_1 = F_3^* \otimes \bigwedge^2 F_1$$

The positive part of the Lie algebra has one component $\underline{g}_1 = F_3^* \otimes \bigwedge^2 F_1$, *so it is the space of skew-symmetric matrices.*

Three critical representations are (see [LW])

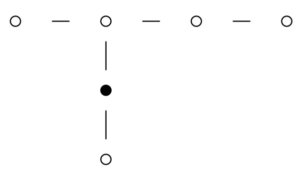
$$W(d_3) = F_2^* \otimes [\oplus_{k\geq 0} S_{k-1}F_3^* \otimes \bigwedge^{2k} F_1],$$
$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes F_1],$$
$$W(a_2) = \mathbf{C} \otimes [\oplus_{k\geq 0} S_k F_3^* \otimes \bigwedge^{2k+1} F_1].$$

The corresponding fundamental representations of $\underline{g}(T_{p,q,r})$ *are the two half-spinor representations (first and third) and the vector representation (second).*

3.6 Proposition. *Consider the format* $(1,5,6,2)$. *These are complexes*

$$0 \to R^2 \to R^6 \to R^5 \to R.$$

The graph $T_{p,q,r}$ *is the graph of type* E_6.



The corresponding Lie algebra is

$$\underline{g}(E_6) = \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2,$$

with

$$\underline{g}_0 = \underline{sl}(F_3) \oplus \underline{sl}(F_1) \oplus \mathbf{C}, \underline{g}_1 = F_3^* \otimes \bigwedge^2 F_1, \underline{g}_2 = \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1,$$
$$\underline{g}_{-i} = \underline{g}_i^* \quad for \quad i = 1,2.$$

The critical representations are (see [LW])

$$W(d_3) = F_2^* \otimes [F_3 \oplus \bigwedge^2 F_1 \oplus F_3^* \otimes \bigwedge^4 F_1 \oplus \bigwedge^2 F_3^* \otimes S_{2,1^4} F_1],$$
$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes F_1 \oplus \bigwedge^2 F_3^* \otimes \bigwedge^3 F_1 \oplus S_{2,1}F_3^* \otimes \bigwedge^5 F_1],$$
$$W(a_2) = \mathbf{C} \otimes [F_1 \oplus F_3^* \otimes \bigwedge^3 F_1 \oplus [\bigwedge^2 F_3^* \otimes \bigwedge^4 F_1 \otimes F_1 \oplus S_2 F_3^* \otimes \bigwedge^5 F_1]$$
$$\oplus S_{2,1}F_3^* \otimes S_{2^2,1^3} F_1 \oplus S_{2,2}F_3^* \otimes S_{2^4,1} F_1].$$

Notice the following pattern:

- The first graded components (green) give the maps d_3, d_2, a_2;
- the second graded components (blue) give three components of the multiplicative structure on the resolution $\mathbb{F}_\bullet$;

- and here is the main thing: the last graded components (red) give three differentials in another complex

$$R \to F_1 \to F_2 \to F_3.$$

one can check that the compositions of these maps are indeed zero.

This phenomenon persists for all Dynkin formats of cyclic modules except for $(1, n, n, 1)$ (n odd).

3.7 Proposition. *For each Dynkin format except $(1, n, n, 1)$ (n odd) we have the following pattern for three critical representations:*

(1) *The first graded components give the maps d_3, d_2, a_2;*
(2) *the second graded components give three components of the multiplicative structure on the resolution $\mathbb{F}_\bullet$;*
(3) *and again: the last graded components give three differentials in another complex* **of the same format**, *which is*

$$F_3^* \to F_2 \to F_1^* \to R.$$

for all formats except $(1, 4, n, n-3)$ and $(1, 5, 6, 2)$.
(4) *For the format $(1, 4, n, n-3)$ because of extra duality in the graph, the complex $\mathbb{F}^{top}$ looks a little different, i.e.*

$$\mathbb{F}_\bullet^{top}:\ 0 \to F_3 \to F_2^* \to F_1^* \to R.$$

(5) *For the format $(1, 5, 6, 2)$ because of extra duality in the graph, the complex $\mathbb{F}^{top}$ looks a little different, i.e.*

$$\mathbb{F}_\bullet^{top}:\ 0 \to F_3^* \to F_2^* \to F_1^* \to R.$$

Moreover, the second and third differential get switched, i.e., $W(d_3)^{top}$ gives the second differential and vice versa.

We call this complex $\mathbb{F}_\bullet^{top}$.

4. Perfect ideals of codimension 3.

In view of previous section it is natural to ask whether the Dynkin formats play a special role in classifying **perfect ideals** of codimension 3. Let us restrict to the rank sequences $(1, r_2, r_3)$ i.e., resolutions of cyclic modules. We have an equivalence relation of **algebraic linkage**. It is an equivalence relation $\sim$ on perfect ideals of codimension 3 generated by

$$I \sim J := (x_1, x_2, x_3) : I$$

where (x_1, x_2, x_3) is a regular sequence from I,

Let us assume additionally that (x_1, x_2, x_3) were chosen from a minimal set of generators of I. Then, if R/I has a resolution

$$0 \to R^{r-1} \to R^{r+q} \to R^{q+2} \to R$$

then R/J has resolution

$$0 \to R^{q-1} \to R^{r+q} \to R^{r+2} \to R$$

coming from the dual of the mapping cone of a chain map from the Koszul complex on x_1, x_2, x_3 to the resolution lifting the natural surjection $R/(x_1, x_2, x_3) \longrightarrow R/I$. This symmetry corresponds to flipping two arms of the $T_{p,q,r}$ graph.

4.1 Conjecture. (**The LICCI Conjecture**). The format is Dynkin if and only if every ideal with resolution of this format is linked to a complete intersection.

We can prove (jointly with L. Christensen, O. Veliche, [CVW2]) that if a format is not Dynkin, there is an ideal with resolution of this format which is not LICCI, We can prove that for Dynkin format the Huneke-Ulrich (see [HU]) obstruction does not happen.

We want to use the generic ring $\hat{R}_{gen}$ to produce perfect ideals I such that R/I has a minimal free resolution of a Dynkin format.

It is therefore natural to ask which points in $\mathrm{Spec}(\hat{R}_{gen})$ correspond to resolutions of Cohen-Macaulay modules. We denote the open set of such points by U_{CM}. Let us restrict to the rank sequences $(1, r_2, r_3)$ i.e., resolutions of cyclic modules The ring R_a is generated by three representations d_3, d_2 and a_2 and a_1. We do not list a_1 later, as it is a trivial representation and we see it also in $\hat{R}_{gen}$, so it is just an extra variable in both rings R_a and $\hat{R}_{gen}$. This means $\hat{R}_{gen}$ should be generated by three representations $W(d_3)$, $W(d_2)$, $W(a_2)$. As we have seen each representation W is graded $W = W_0 \oplus W_1 \oplus \cdots$. If $T_{p,q,r}$ is Dynkin, we have

$$W = W_0 \oplus W_1 \oplus \cdots W_{top}$$

as the critical representations are finite dimensional.

Assume we have a Dynkin format different than $(1, n, n, 1)$ with n odd, $(1, 4, n, n-3)$ and $(1, 5, 6, 2)$ Three tensors from the top components of $W(d_3)$, $W(d_2)$, $W(d_1)$ can be arranged into differentials of a complex

$$\mathbb{F}^{top}_\bullet : \ 0 \to F_3^* \to F_2 \to F_1^* \to R.$$

For the format $(1, 4, n, n-3)$ because of extra duality in the graph, the complex $\mathbb{F}^{top}$ looks a little different, i.e.

$$\mathbb{F}^{top}_\bullet : \ 0 \to F_3 \to F_2^* \to F_1^* \to R.$$

For the format $(1, 5, 6, 2)$ because of extra duality in the graph, the complex $\mathbb{F}^{top}$ looks a little different, i.e.

$$\mathbb{F}^{top}_\bullet : \ 0 \to F_3^* \to F_2^* \to F_1^* \to R.$$

Moreover, the second and third differential get switched.

We have the following conjecture.

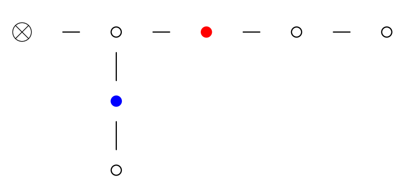
4.2 Conjecture. (**The $U_{CM} = U_{split}$ Conjecture**). A point in $\mathrm{Spec}(\hat{R}_{gen})$ (i.e., a homomorphism $\hat{R}_{gen} \to S$ to a local ring S) corresponds to a resolution of perfect ideal if and only if the localization of the complex $\mathbb{F}^{top}_\bullet$ is split exact at that point.

Let us call the open set on which $\mathbb{F}^{top}_\bullet$ is split exact U_{split}. So the conjecture means that $U_{CM} = U_{split}$.

To see that $U_{split} \subset U_{CM}$ we need to see what is the form of the complex $\mathbb{F}^{gen}_\bullet$ on points from U_{split}. Here we can use the structure of the $\underline{g}(T_{p,q,r})$ representation on $\hat{R}_{gen}$. Using the highest-lowest weight symmetry between W_0 and W_{top} we do the reverse calculation: assume that the original complex $\mathbb{F}^{gen}_\bullet$ is split exact, reduce it to the canonical form and calculate in the resulting affine space on defect variables the differentials of the complex $\mathbb{F}^{top}_\bullet$.

It turns out we get nice perfect ideals with resolutions of corresponding format. They are the defining ideals of certain Schubert varieties.

The pattern comes from the $T_{p,q,r}$ graph. We do the example of graph E_7 but it is the same for other Dynkin formats.



The two bullets denote the nodes corresponding to two linked formats of type E_7

$$0 \to R^3 \to R^7 \to R^5 \to R$$

$$0 \to R^2 \to R^7 \to R^6 \to R$$

The node $\otimes$ shows the maximal parabolic we need to take and we will denote by c the central node.

Denote by s_i the simple reflection in the Weyl group corresponding to node i. In particular, s_c will be the simple reflection corresponding to central node. We have two elements w_1, w_2 of length 3 in W: $w_1 = s_{\bullet} s_c s_{\otimes}$ and $w_2 = s_{\bullet} s_c s_{\otimes}$. We intersect two Schubert varieties X_{w_1}, X_{w_2} corresponding to these Weyl group elements with the the opposite big cell of $G/P_{\otimes}$ to get two affine varieties Y_{w_1}, Y_{w_2} with ideals I_{w_1}, I_{w_2} in the polynomial ring R which is the coordinate ring of the opposite big cell in $G/P_{\otimes}$. The R-modules R/I_{w_1} and R/I_{w_2} have resolutions of exactly needed Dynkin formats! Moreover, the ideals I_{w_1}, I_{w_2} are linked in the big opposite cell of this $G/P_{\otimes}$.

They are linked by a regular sequence given by three Plücker coordinates that are in both ideals, which are the extremal Plücker coordinates corresponding to Weyl group elements $id, s_{\otimes}, s_c s_{\otimes}$.

4.3 Proposition. *Type D_n. The opposite big cell can be identified with $n \times n$ skew-symmetric matrices.*

The variety Y_{w_2} is given by submaximal Pfaffians of odd shaped skew-symmetric matrix (if n is even we have to cut out the first row and column).

The variety Y_{w_1} is an almost complete intersection, i.e., its ideal has 4 generators, it is given by an $n \times n$ skew-symmetric matrix and 3 n-vectors (see [CVW1] for precise description). This situation was already described by Lucho Avramov in 1981 and by Anne Brown ([Br]) in 1987.

4.4 Proposition. *Type E_6. We get two Schubert varieties with resolutions*

$$0 \to R^2(-7) \to R^6(-5) \to R(-4) \oplus R^4(-3) \to R.$$

They (or, rather, their linear sections) were described in [CJKW]. There is an equivariant form of these ideals. One of them are contained in the representation $\bigwedge^3 F$, $\dim F = 6$. The generators come from an SL_6 invariant Δ of degree 4 on $\bigwedge^3 F$) and its partial derivatives with respect to variables x_{456}, x_{356}, x_{256} and x_{156}. The other has generators Δ and its partials with respect to x_{456}, x_{356}, x_{346} and x_{345}. They are linked by a regular sequence $(\Delta, \partial\Delta/\partial x_{456}, \partial\Delta/\partial x_{356})$.

4.5 Proposition. *Type E_7. The variety Y_{w_1} has a resolution in the graded format*

$$0 \to R^3(-13) \to R^7(-10) \to R(-7) \oplus R^4(-6) \to R.$$

Its linear section is contained in the representation $\bigwedge^3 F$ where $\dim F = 7$. The generators come from an SL_7 invariant Δ of degree 7 on $\bigwedge^3 F$ ($\dim F = 7$) and its partial derivatives with respect to x_{567}, x_{467}, x_{457}, x_{456}. The variety Y_{w_2} has a resolution of graded format

$$0 \to R^2(-13) \to R^7(-9) \to R(-7) \oplus R^5(-6) \to R.$$

Its linear section is also contained in the representation $\bigwedge^3 F$. The generators come from Δ and its partial derivatives with respect to x_{567}, x_{467}, x_{367}, x_{267}, x_{167}.

They are linked by a regular sequence $(\Delta, \partial\Delta/\partial x_{567}, \partial\Delta/\partial x_{467})$.

4.6 Remark. These ideals are small enough so these resolutions were handled by Macaulay2, so we can see the differentials!

4.7 Proposition. *Type E_8. The variety Y_{w_1} has a resolution in the graded format*

$$0 \to R^4(-31) \to R^8(-25) \to R(-16) \oplus R^4(-15) \to R.$$

It is contained in the representation $\bigwedge^3 F$ where $\dim F = 8$. The generators of its linear section comes (conjecturally as we cannot verify the Schubert variety intersects the appropriate hyperplane in the right dimension) from SL_8 invariant Δ of degree 16 on $\bigwedge^3 F$, $\dim F = 8$, and its partial derivatives with respect to variables x_{678}, x_{578}, x_{568}, x_{567}. The variety Y_{w_2} has a resolution of graded format

$$0 \to R^2(-31) \to R^8(-21) \to R(-16) \oplus R^6(-15) \to R.$$

Its linear section is also (conjecturally) contained in the representation $\bigwedge^3 F$ and its defining ideal is generated by Δ and its partials with respect to x_{678}, x_{578}, x_{478}, x_{378}, x_{278}, x_{178}. They are linked by a regular sequence $(\Delta, \partial\Delta/\partial x_{678}, \partial\Delta/\partial x_{578})$.

4.8 Remark. These resolutions are so big they have not been handled by Macaulay2.

So the main questions are:

4.9 Questions.

(1) One knows that $U_{split} \subset U_{CM}$. Do we have $U_{CM} = U_{split}$?
(2) **Genericity Conjecture**: Prove that the split form indeed gives a generic perfect ideal of a given format.

5. Beyond the Dynkin types

This material is for the future, probably not for this workshop, I just wanted to indicate the story does not end with Dynkin formats.

5.1 Proposition. *The pattern with two opposite Schubert varieties Y_{w_1}, Y_{w_2} of codimension 3 is true for any format (with $p = 2$). In the big open cell we have* **the opposite Schubert varieties** *which have* **infinite dimension** *and* **finite codimension** *in non-Dynkin cases. The opposite Schubert varieties are Cohen-Macaulay by a result of Kashiwara and Shimozono, (see[KS]).*

5.2 Questions. One has the following natural questions regarding opposite Schubert varieties.

(1) Do the ideals of these opposite Schubert varieties of codimension 3 have free resolutions of the desired format over the coordinate ring R of some big open cell?
(2) There is no big opposite cell here, but one can hope to find a sequence of big open cells Y_n, so one could possibly produce from this approach a sequence of Noetherian rings R_n and resolutions of the corresponding format over R_n that will be universal in the sense that, for each family of resolutions $\mathbf{G}_\bullet$ of that format over a Noetherian ring R **with bounded regularity**, they all will come from some R_n.

5.3 Example. Another possible generalization of the linear sections in $\bigwedge^3 F$ for $dim\ F = 6, 7, 8$ above is as follows. Consider $\bigwedge^r F$ with $dim\ F = n$. Then we have a specific $SL(F)$ invariant: a hyperdiscriminant Δ. I expect that there are two ideals I_{w_1} and I_{w_2} defined as follows: The ideal I_{w_1} is generated by Δ and its partials with respect to $x_{i,n-r+2,n-r+3,\dots,n}$ for $1 \le i \le n-r+1$. I expect that the module R/I_{w_1} has the resolution

$$0 \to R^{r-1} \to R^n \to R^{n-r+2} \to R.$$

The ideal I_{w_2} is generated by Δ and its partials with respect to

$$x_{\{n-r,n-r+1,n-r+2,n-r+3,\dots,n\}\setminus i} \ \text{ for } \ n-r \le i \le n.$$

I expect that the module R/I_{w_1} has the resolution

$$0 \to R^{n-r-1} \to R^n \to R^{r+2} \to R.$$

They are minimally linked to each other via the regular sequence consisting of

$$\Delta, \partial\Delta/\partial x_{n-r+1,n-r+2,n-r+3,\dots,n}, \partial\Delta/\partial x_{n-r,n-r+2,n-r+3,\dots,n}.$$

Of course I do not expect these to be a generic resolutions of given formats, but it could be one of the members of the hierarchy one expects.

5.4 Questions. Here are some final questions

(1) What happens in extended Dynkin cases? For the format

$$0 \to R^3 \to R^8 \to R^6 \to R$$

we have two natural families of resolutions: for the ideal of 2×2 minors of a 2×4 generic matrix, and for the ideal of 2×2 minors of a 3×3 generic symmetric matrix.

(2) How do we distinguish these families using our approach?

(3) What about $n > 3$?

6. Gorenstein ideals of codimension 4

Here one has some important progress in recent years. The main new idea is the existence of the spinor structure on a resolution of a Gorenstein ideal of codimension 4. The original idea of such structure comes from Miles Reid [R]. In [CJW] the authors interpreted the spinor coordinates as square roots of certain Buchsbaum-Eisenbud multipliers.

Examples show that there is a marked difference between Gorenstein ideals with 6,7,8 generators and the ones with more generators. This might suggest that the cases of 6,7,8 generators might be easier to classify than with more generators (akin to Dynkin formats vs. non-Dynkin formats for perfect ideals of codimension 3). The arguments for this are two-fold (see two first points below).

Below R denotes a local Noetherian ring.

(1) Doubling construction. If I is a perfect ideal of codimension 3 in R with minimal free resolution of R/I being the complex $\mathbf{F}_\bullet$, then the minnimal free resolution of the canonical module $\omega_{R/I}$ is the dual complex $\mathbf{F}\bullet^*$. Under mild assumptions $\omega_{R/I}$ is an ideal in R/I. Then the cokernel of the injection $\omega_{R/I} \to R/I$ is a module R/J where J is Gorenstein of codimension 4. If R/J has ≤ 8 generators, then R/I has to have a resolution of Dynkin format, otherwise it could happen it does not.

(2) Spinor coordinates. If we consider a minimal free resolution of R/J where J is a Gorenstein ideal of codimension 4, one expects the spinor coordinates to be inside of J (it is true if J is a radical ideal). It is an interesting question to prove it in general. The coefficients of linear combinations expressing spinor coordinates in terms of generators of the ideal should be the key for classification. In all known examples with 6,7,8 generators at least one of spinor coordinates is among minimal generators of J. In the case of 9 generators we have 2×2 minors of 3×3 generic matrix and the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. In both cases the spinor coordinates have degree 3 so they are not among minimal generators of J.

(3) In each of our Schubert examples the sum of ideals I_{w_1} and I_{w_2} defines a Gorenstein ideal J of codimension 4. In each case there is exactly one spinor coordinate that is a minimal generator of our ideal J: the invariant Δ (i.e., the first Plücker coordinate)..

Another point is an interesting analogy. For perfect ideals of codimension 3 the resolutions of Dynkin formats are scarce among examples coming from geometry. But the extended Dynkin formats have well-known examples (Eagon-Northcott complex of 2×2 minors of a generic 2×4 matrix and 2×2 minors of a generic symmetric 3×3 matrix. Both have format $(1, 6, 8, 3)$.

Similarly for Gorenstein ideals: for 6,7,8 generators the examples are scarce (for 6 generators one expects all are just hyperplane sections in Gorenstein ideals of codimension 3), but for 9 generators there are two "generic examples: 2×2 minors of 3×3 generic matrix and the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, which can be thought of as $2 \times 2 \times 2$ minors of $2 \times 2 \times 2$ generic matrix. It is a curious coincidence that both these examples are doublings of the codimension 3 perfect ideals mentioned above: The ideal of 2×2 minors of 3×3 generic matrix is a doubling of 2×2 minors 3×3 generic symmetric matrix (just symmetrize the matrix; one needs to assume that $\frac{1}{2}$ is in our local ring R). The ideal of $2 \times 2 \times 2$ minors of $2 \times 2 \times 2$ generic matrix is a doubling of 2×2 minors of 2×4 generic matrix (just flatten the $2 \times 2 \times 2$ matrix to get a 2×4 matrix).

References:

- [AKM] L. Avramov, A. Kustin, M. Miller, Poincaré series of modules over local rings of small embedding codepth or small linking number, J. Algebra **118** (1988), no. 1, 162–204
- [BE1] D. Buchsbaum, D. Eisenbud, Some structure theorems for finite free resolutions, Adv. in Math., **1**(1974), 84–139,
- [BE2] D. Buchsbaum, D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. **99** (1977), no. 3, 447–485.
- [Br] A. Brown, A structure theorem for a class of grade three perfect ideals, J. Algebra. **105** (1977), no. 3, 308–327.
- [CJKW] E. Celikbas, J. Laxmi, W. Kraśkiewicz, J. Weyman, The family of perfect ideals of codimensionn 3, of type 2 with 5 generators, arXiv 1812.09736,
- [CJW] E. Celikbas, J. Laxmi, J. Weyman, Spinor structures on free resolutions of Gorenstein ideals of codimension four, arXiv 1912.07510,
- [CVW1] L. Christensen, O. Veliche, J. Weyman, On the almost complete intersections of grade 3.
- [CVW2] L. Christensen, O. Veliche, J. Weyman, Free resolutions of Dynkin format and the licci property of grade 3 perfect ideals,
- [H] M. Hochster, , CBMS Regional Conference Series in Mathematics vol. 24, 1975.
- HU C. Huneke, B. Ulrich, The structure of linkage, Annals of Mathematics, (2) 126 (1987), No.2, pp.277-334,
- [KS] M. Kashiwara, M. Shimozono, Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials, Duke Math. J. **148** (2009).
- [LW] K-H. Lee, J. Weyman, Some branching rules for Kac-Moody Lie algebras, Communications of the Korean Mathematical Society, vol. 34, no. 4, 1079-1098.
- [PW] P. Pragacz, J. Weyman, On the Generic Free Resolutions, Journal of Algebra, 128, No. 1 (1990), 1-44.
- [R] M. Reid, Gorenstein of codimension 4 - a general structure theory, available from https://homepages.warwick.ac.uk/∼masda/
- [SW20] S. Sam, J. Weyman, Schubert varieties and finite free resolutions of length three.
- [W] J. Weyman, Generic Free Resolutions and Root Systems, Annales de l'Institute Fourier, 68(3), 2018, p. 1241-1296, arXiv 1609.02083.
- [W1] J. Weyman, On the Structure of Resolutions of Length 3, Journal of Algebra, 126 No.1 (1989), 1-33.

University of Connecticut