NOTES ON FINITE FREE RESOLUTIONS

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Abstract. These are the notes for the ICERM workshop on Finite Free Resolutions and Representation Theory from August 2020. They are based on similar notes for the similar workshop held at UCSD in August 2019. I give an introduction to the recent progress on finite free resolutions resulting from the ideas of the paper [32]. We start with the basic commutative algebra results on acyclicity of complexes and the structure of finite free resolutions. Then we proceed to some examples of resolutions of small formats. We also discuss the situation for perfect ideals of codimension 3. Finally we look at some examples of ideals in the polynomial ring of three variables.

1. Introduction

In the paper [32] I constructed specific generic rings $\hat{R}_{gen}$ for resolutions of length three of all formats. The structure of these generic rings is related to T-shaped graphs $T_{p,q,r}$ where $(p, q, r) = (r_1+1, r_2-1, r_3+1)$ where (r_1, r_2, r_3) are the ranks of differentials in our complex. I also gave sets of generators for these rings. The main consequence of the construction was that the generic ring $\hat{R}_{gen}$ is Noetherian if and only if the associated graph $T_{p,q,r}$ is Dynkin. Thus we can talk of Dynkin formats and we expect the structure of resolutions to be particularly simple for Dynkin formats.

These notes provide an introduction to the ideas leading to the results of [32]. I tried to do things that can be understood without the knowledge of representations of Kac-Moody Lie algebras, looking at formats corresponding to orthogonal Lie algebra.

The structure of the notes is as follows. The first four sections give the basic material from commutative algebra. Next we give a definition of a generic ring, the account of Hochster's solution of the case $n = 2$, and the description of the generic rings for the formats $(1, n, n, 1)$ and $(1, 4, n, n-3)$.

The remaining sections contain the examples for Dynkin formats. We give explicit description of the generic rings for these formats, describe the Zariski open sets U_{CM} where the generic resolution is a resolution of a perfect module.

Finally we give examples from algebraic geometry (Artinian algebras, points in $\mathbb{P}^2$, $\mathbb{P}^3$, curves in $\mathbb{P}^3$, $\mathbb{P}^4$, surfaces in $\mathbb{P}^4$, $\mathbb{P}^5$) of the occurrence of resolutions of Dynkin formats.

I would like to thank Claudia Miller for looking carefully through the UCSD notes and suggesting many improvements.

2. Buchsbaum-Eisenbud Acyclicity Criterion and Peskine-Szpiro Acyclicity Lemma.

We employ the following notation regarding finite free resolutions. In principle all rings we encounter are Noetherian unless otherwise stated. We will consider the complexes of free modules over a Noetherian ring R. By definition these are sequences of homomorphisms

$$\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \to F_1 \xrightarrow{d_1} F_0$$

where $F_i = R^{f_i}$ and $d_{i-1}d_i = 0$ for $i = 2, \ldots, n$. We say that $\mathbb{F}_\bullet$ is *a finite free resolution* or it is *acyclic* if the only non-zero homology module of $\mathbb{F}_\bullet$ is $H_0(\mathbb{F}_\bullet) = M$. In this case $\mathbb{F}_\bullet$ is a finite free resolution of the R-module M.

After choosing bases on each free module F_i we can think of d_i as an $f_{i-1} \times f_i$ matrix. We denote by *rank* r_i of the linear map d_i the maximal size of non-vanishing minor of d_i. The ideals $I(d_i) := I_{r_i}(d_i)$ generated by all minors of d_i of size r_i are essential for our approach.

Before we start we need some properties of the ideals $I(d_i)$.

Lemma 2.1. *Let $d : F \to G$ be a map of free R-modules.*

(1) The ideal $I_r(d)$ generated by $r \times r$ minors of the matrix of d does not depend on the choice of basis in F and G,

(2) Assume R is local, d has rank r and that $I(d) = R$. Then after change of bases in F and G the matrix of the map d can be brought to the form

$$\begin{bmatrix} 1_r & 0 \\ 0 & 0 \end{bmatrix}$$

where 1_r is an $r \times r$ identity matrix.

We have two criteria for the acyclicity of $\mathbb{F}_\bullet$.

Theorem 2.2. *(Buchsbaum-Eisenbud,* [6]*) The complex $\mathbb{F}_\bullet$ is acyclic if and only if the following two conditions hold*

(1) $f_i = r_i + r_{i+1}$ for all $1 \le i \le n$, with the convention that $r_{n+1} = 0$.

(2) For all $1 \le i \le n$ we have $\mathrm{depth}(I(d_i)) \ge i$.

Remark 2.1. (1) The assumption that the map d_n is injective is essential. Otherwise we have the examples similar to the following. Take $R = K[X]/(X^2)$. Take the complex with $F_i = R$, $d_i = (X)$. We get a complex which is acyclic but $\mathrm{rank}(d_i) + \mathrm{rank}(d_{i+1}) > f_i$ for all i.

(2) This statement is true over non-Noetherian rings with the appropriate definition of grade. This is explained in the book [22] of Northcott. He defined the true grade as

$$\mathrm{Grade}_R(I, M) = \sup_{n \ge 0} \{\mathrm{grade}_{R[x_1,\ldots,x_n]}(I \otimes_R R[x_1, \ldots, x_n], M \otimes_R R[x_1, \ldots, x_n])\}$$

where grade is defined as the maximal length of a regular sequence on M contained in I, and proved that Buchsbaum-Eisenbud acyclicity criterion holds with this definition of the depth over any ring R. Thus we will use the theory over arbitrary rings and in case of (possibly) non-Noetherian ring, depth will mean the true grade in the above sense.

There is another statement which is almost equivalent.

Theorem 2.3. *(Lemme d'Acyclicite, Peskine-Szpiro* [23]*). Let $\mathbb{F}_\bullet$ be a free complex. Then* $\mathbf{F}_\bullet$ *is acyclic if and only if $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic at all prime ideals P such that* $\mathrm{depth}\, PR_P < n$.

Proof. We just need to prove that if $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic for all prime ideals P such that depth $PR_P < n$, then $\mathbf{F}_\bullet$ is acyclic. We prove that for every prime P the complex $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic. We do it by induction on $\dim R_P$. If $\dim R_P < n$ then depth $PR_P \leq \dim R_P < n$ and we are done by assumption. So let us assume that $\dim R_P \geq n$. Then we have a complex $\mathbb{F}_\bullet$ of length $n \leq \operatorname{depth} \mathfrak{m}$ over a local ring $(S, \mathfrak{m})$ such that for every prime ideal $P \neq \mathfrak{m}$, the complex $\mathbb{F}_\bullet \otimes_S S_P$ is acyclic. The result then follows from the lemma.

Lemma 2.4. *Let S be a commutative Noetherian ring, $I \subset S$ an ideal. Assume that we have a complex*

$$\mathbb{M}_\bullet : 0 \to M_n \to M_{n-1} \to \ldots \to M_1 \to M_0$$

of finitely generated S-modules. Denote by H_k the k-th homology module of the complex $\mathbb{M}_\bullet$. Assume that

(1) $\operatorname{depth}(I, M_k) \geq k$ *for* $1 \leq k \leq n$,
(2) $\operatorname{depth}(I, H_k) = 0$ *for* $1 \leq k \leq n$.

Then the complex $\mathbb{M}_\bullet$ is acyclic.

Apply Lemma 2.4 to the complex $\mathbb{F}$ over S and $I = \mathfrak{m}$. Indeed, $\operatorname{depth}(I, F_k) = n \geq k$ for $k = 0, \ldots, n$. Also, for every $P \neq \mathfrak{m}$ $(H_k)_P = 0$ which means the only associated prime of M is $\mathfrak{m}$, so $\operatorname{depth}(\mathfrak{m}, H_k) = 0$ for $k = 1, \ldots, n$. It remains to prove the lemma.

Let B_k denote the module of boundaries in degree k, Z_k-the modules of cycles in degree k. Let us take the biggest m such that $B_m \neq Z_m$. We will prove $m = 0$. We make the following claims.

(1) $\operatorname{depth}(I, B_m) \geq m + 1 \geq 2$,
(2) $\operatorname{depth}(I, M_m) \geq m \geq 1$.

Considering the short exact sequence

$$0 \to B_m \to Z_m \to H_m \to 0$$

and the long exact sequence of Ext's, noting that $\operatorname{depth}(I, Z_m) \geq 1$ because $Z_m \subset M_m$, we get

$$\ldots 0 = \operatorname{Ext}^0(R/I, Z_m) \to \operatorname{Ext}^0(R/I, H_m) \to 0$$

which means that $\operatorname{depth}(I, H_m) \geq 1$, a contradiction.

Since the second claim is in the assumption of the lemma, it is enough to prove the first one. We use the exact sequence

$$0 \to M_n \to \ldots \to M_{m+1} \to B_m \to 0,$$

we divide it into short exact sequences and use the long exact sequences of Ext's. □

Now we are ready to prove Theorem 2.2. First we prove that the rank and depth conditions are sufficient. We use Theorem 2.3. So it is enough to show that $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic for depth $PR_P < n$. But this implies that $I(d_n)_P$ is a unit ideal. This, by row reduction implies that the complex $\mathbb{F}_\bullet \otimes_R R_P$ can be written as a direct sum of the isomorphism $R_P^{f_n} \to R_P^{f_n}$ and the free complex of length $n-1$ of format $(f_0, \ldots, f_{n-2}, r_{n-1})$ satisfying the conditions of Theorem 2.2. By induction on n we see that this complex is indeed acyclic, so we are done.

Finally we prove that rank and depth conditions are necessary. We use induction on n. First we treat the case $n = 1$. We have a map $d_1 : F_1 \to F_0$ and we need to show that if d_1 is injective then $f_1 \le f_0$ and $I(d_1)$ contains a non-zerodivisor. We note that if d_i is injective then the f_1-st exterior power of d_1 is also injective. Indeed, we have the commutative diagram

$$\begin{array}{ccc} F_1^{\otimes f_1} & \to & F_0^{\otimes f_1} \\ \uparrow & & \uparrow \\ \bigwedge^{f_1} F_1 & \to & \bigwedge^{f_1} F_0 \end{array}$$

where the horizontal maps are induced by d_1 and vertical are the natural injections. It follows that the map $\bigwedge^{f_1} F_1 \to \bigwedge^{f_1} F_0$ has to be injective which gives the rank and the depth condition for $n = 1$.

To prove that the conditions are necessary for $n > 1$ one needs to take the following steps. One takes the multiplicative subset S of non-zerodivisors and then one knows that natural map $f : R \to S^{-1}R$ is injective. So if $\mathbb{F}_\bullet$ is acyclic, then $\mathbb{F}'_\bullet := \mathbb{F}_\bullet \otimes_R (S^{-1}R)$ is acyclic. We denote $d'_i = S^{-1}d_i$. We have $\operatorname{rank} d'_i = \operatorname{rank} d_i$ because the map f is injective and localization commutes with taking exterior powers of maps of free modules. This means $I(d'_i) = S^{-1}I(d_i)$. We know by the case $n = 1$ that $I(d_n)$ contains a non-zero divisor. This means that $I(d'_i) = S^{-1}R$.

Now we have the following lemma

Lemma 2.5. *The module* $\operatorname{Coker}(d'_n)$ *is free over* $S^{-1}R$.

Proof. This follows from the general fact that a projective module of constant rank over a semilocal ring is free (Bourbaki, "Commutative Algebra" Chapter 2, Section 5, Proposition 5). So we need to show that $\operatorname{Coker}(d'_n)$ is projective of constant rank. Therefore it is enough to show that for every prime ideal P in $S^{-1}R$, over a local ring R_P $\operatorname{Coker}(d_n \otimes_R R_P)$ is free of rank $\operatorname{rank} F_n - \operatorname{rank}(d_n)$. But this is clear by 2.1. □

So we reduced the length of our complex, because we need to prove the result for the complex

$$0 \to \operatorname{Coker}(d'_n) \overset{d''_{n-1}}{\to} F'_{n-2} \to \ldots \to F'_1 \to F'_0.$$

Note that $I(d''_{n-1}) = I(d'_{n-1})$, it follows from row reduction. We proceed by descending induction on k, we need to show that $I(d'_k) = S^{-1}R$. But if we take the biggest k such that $I(d'_k) \ne S^{-1}R$, then all the maps d_l for $l > k$ split so we are reduced to the case of the injective map. Then from the case $n = 1$ we see that $I(d'_k)$ has to contain a non-zerodivisor, but in $S^{-1}R$ every non-zerodivisor is a unit, which gives a contradiction.

Splitting all the maps d'_k shows that the rank conditions $r_i + r_{i+1} = \operatorname{rank} F_i$ have to be satisfied.

To check the depth conditions, assume that $\mathbb{F}_\bullet$ is acyclic but $\operatorname{depth} I(d_k) = l < k$ for some k. Let us take the biggest such k. By the standard commutative algebra argument there exists a prime ideal P, $I(d_k) \subset P$ such that $\operatorname{depth} PR_P = l$. Let us localize at P. Then for all $m > k$ $I(d_m) \otimes_R R_P = R_P$ (because their depth over R was $\ge m$), so they split. However we have that $I(d_k) \otimes_R R_P \ne R_P$. This means that the projective dimension of $H_0(\mathbb{F}_\bullet) \otimes_R R_P$ is equal to k. But since $\operatorname{depth} PR_P = l < k$, we get a contradiction with the

Auslander-Buchsbaum formula

$$\mathrm{pd}_{R_P} M + \mathrm{depth}(PR_P, M) = \mathrm{depth}\, PR_P.$$

3. The First and Second Structure Theorems of Buchsbaum and Eisenbud.

Throughout we will employ the following notation. We will deal with finite free resolutions

$$\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \to F_1 \xrightarrow{d_1} F_0$$

with $\mathrm{rank}(d_i) = r_i$, $F_i = R^{f_i}$ and we will assume $f_i = r_i + r_{i+1}$ for $1 \le i \le n$. We refer to the sequence $(f_0, f_1, \ldots, f_n)$ as *the format* of the resolution $\mathbb{F}_\bullet$.

We have the following First Structure Theorem of Buchsbaum and Eisenbud ([7], Theorem 3.1).

Theorem 3.1. *We have the unique sequence of maps*

$$a_i : \otimes_{j=i}^n (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}} \to \bigwedge^{r_i} F_{i-1}$$

such that

(1) $a_n : \bigwedge^{r_n} F_n \to \bigwedge^{r_n} F_{n-1}$ *is just* $\bigwedge^{r_n} d_n$,

(2) We have a commutative diagram

$$\begin{array}{ccc} \bigwedge^{r_i} F_i & \xrightarrow{\bigwedge^{r_i} d_i} & \bigwedge^{r_i} F_{i-1} \\ \downarrow = & & \uparrow a_i \\ \bigwedge^{f_i} F_i \otimes \bigwedge^{r_{i+1}} F_i^* & \xrightarrow{\bigwedge^{f_i} F_i \otimes a_{i+1}^*} & \otimes_{j=i}^n (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}} \end{array}$$

Proof. In the paper [7] the theorem is stated in the SL-equivariant form. More precisely, one claims the isomorphisms $\bigwedge^{r_i} F_i \equiv \bigwedge^{r_{i+1}} F_i^*$ and one claims the commutativity of the diagram

$$\begin{array}{ccc} \bigwedge^{r_i} F_i & \xrightarrow{\bigwedge^{r_i} d_i} & \bigwedge^{r_i} F_{i-1} \\ \downarrow = & & \uparrow a_i \\ \bigwedge^{r_{i+1}} F_i^* & \xrightarrow{a_{i+1}^*} & R \end{array}$$

Since the claim follows from certain factorization, both versions are equivalent (once we choose bases in the modules F_i).

The idea of the proof is very simple. Assume that the map a_i was constructed and $i \ge 2$. We will construct the map a_{i-1}. We consider the map

$$\tilde{d}_i : \bigwedge^{r_i} F_{i-1} \to \bigwedge^{r_i+1} F_{i-1} \otimes F_i^*$$

induced by the differential d_i treated as a map in $d_i : R \to F_{i-1} \otimes F_i^*$.

Then one considers the complex

$$0 \to R \xrightarrow{a_i} \bigwedge^{r_i} F_{i-1} \xrightarrow{\tilde{d}_i} \bigwedge^{r_i+1} F_{i-1} \otimes F_i^*$$

We claim this complex is acyclic. It is clear that the Fitting ideal of the differential on the right is the ideal $I(d_i)$. It is clear that $I(a_i)$ has depth ≥ 2 because $i \geq 2$ and $I(d_i) = I(a_{i+1})I(a_i)$. So one needs to prove that our complex satisfies the rank conditions. But this can be done after localizing, i.e., one can assume $\mathbb{F}_\bullet$ is split exact. In this case it is an exercise for the reader.

We also see that the composition

$$\bigwedge^{r_{i-1}} F^*_{i-2} \overset{\bigwedge^{r_{i-1}} d^*_{i-1}}{\rightarrow} \bigwedge^{r_{i-1}} F^*_{i-1} = \bigwedge^{r_i} F_{i-1} \overset{\tilde{d_i}}{\rightarrow} \bigwedge^{r_i+1} F_{i-1} \otimes F_i^*$$

is zero. From this it follows that there exists a map a making the diagram

$$\begin{array}{ccc}
\bigwedge^{r_{i-1}} F^*_{i-2} & \xrightarrow{\bigwedge^{r_{i-1}} d^*_{i-1}} & \bigwedge^{r_{i-1}} F^*_{i-1} \\
 & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle a_i} & \\
 & R &
\end{array}$$

commute. Now we can take $a_{i-1} = a^*$. □

The main example which was the motivation for that theorem is the case of $n = 2$, the format $(1, n, n-1)$.

Theorem 3.2. *(Hilbert-Burch) Let R be a Noetherian ring and let us assume we have an acyclic complex*

$$0 \rightarrow R^{n-1} \overset{d_2}{\rightarrow} R^n \overset{d_1}{\rightarrow} R.$$

Let us choose basis in our free modules so we can identify d_2 with the $n \times (n-1)$ matrix

$$d_2 = (y_{i,j})$$

and we can identify d_1 with $1 \times n$ matrix $(x_1, \ldots, x_n)$. The there exists a non-zero divisor $a \in R$ such that

$$x_i = (-1)^i a \Delta_i$$

where Δ_i is the determinant of the matrix d_2 with the i-th row omitted.

Buchsbaum and Eisenbud proved also in [7] the Second Structure Theorem which shows how the submaximal exterior power $\bigwedge^{r_i-1} d_i$ factors through F_i^*.

Theorem 3.3. *Let $i \geq 2$. We have a map $b_i : \otimes_{j=i}^n (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}} \otimes F_i^* \rightarrow \bigwedge^{r_i-1} F_{i-1}$ such that the following diagram commutes*

$$\begin{array}{ccc}
\bigwedge^{r_i-1} F_i & \xrightarrow{\bigwedge^{r_i-1} d_i} & \bigwedge^{r_i-1} F_{i-1} \\
\big\downarrow{\scriptstyle =} & & \big\uparrow{\scriptstyle b_i} \\
\bigwedge^{f_i} F_i \otimes \bigwedge^{r_{i+1}+1} F_i^* & \xrightarrow{\bigwedge^{f_i} F_i \otimes (a^*_{i+1})'} & \otimes_{j=i}^n (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}} \otimes F_i^*
\end{array}$$

where $(a^*_{i+1})'$ is the contraction by a^*_{i+1}.

Proof. We show the following facts:

(1) The complex

$$F_i \overset{a'_{i+1}}{\to} \bigwedge^{r_{i+1}+1} F_i \overset{d_{i+1}}{\to} \bigwedge^{r_{i+1}+2} F_i \otimes F_{i+1}^*$$

is exact.

(2) The composition

$$\bigwedge^{r_i-1} F_{i-1}^* \overset{\bigwedge^{r_i-1} d_i^*}{\to} \bigwedge^{r_i-1} F_i^* = \bigwedge^{r_{i+1}+1} F_i \overset{d_{i+1}}{\to} \bigwedge^{r_{i+1}+2} F_i \otimes F_{i+1}^*$$

is zero.

These two facts imply the existence of a factorization b

$$\begin{array}{ccc} \bigwedge^{r_i-1} F_{i-1}^* & \xrightarrow{\bigwedge^{r_i-1} d_i^*} & \bigwedge^{r_i-1} F_{i-1}^* \\ & {\scriptstyle b}\searrow \quad \nearrow{\scriptstyle a'_{i+1}} & \\ & F_i & \end{array}$$

and we take $b_i = b^*$.

The first fact follows by continuing the resolution to the left with the rest of the complex $\mathbb{F}_\bullet$, i.e., considering the complex

$$0 \to F_n \to \ldots \to F_{i+1} \to F_i \overset{a'_{i+1}}{\to} \bigwedge^{r_{i+1}+1} F_i \overset{d_{i+1}}{\to} \bigwedge^{r_{i+1}+2} F_i \otimes F_{i+1}^*$$

and applying the Buchsbaum-Eisenbud acyclicity criterion to it. The second fact one can check on a split complex (after localizing at the multiplicative set S of non-zerodivisors in R). □

Remark 3.1.

(1) The first and second structure theorems of Buchsbaum and Eisenbud are true over non-Noetherian rings with true grade replacing depth. This is proved in the book of Northcott [22].

(2) Bruns proved ([5]) that the first structure theorem is true for complexes acyclic in codimension one, i.e., the complexes $\mathbb{F}_\bullet$ for which $\operatorname{depth}(I(d_i)) \geq 2$ for $i \geq 2$ and $\operatorname{depth}(I(d_1)) \geq 1$.

Remark 3.2.

(1) Let $d : F \to G$ be a map of free R-modules of rank r. The map d induces the map $\bigwedge^r d : R \to \bigwedge^r G \otimes \bigwedge^r F^*$. We also have a map $\tilde{\bigwedge}^r d : G \to \bigwedge^{r+1} G \otimes \bigwedge^r F^*$ induced by $\bigwedge^r d$. Prove that the composition

$$F \overset{d}{\to} G \overset{\tilde{\bigwedge}^r d}{\to} \bigwedge^{r+1} G \otimes \bigwedge^{r} F^*$$

is zero, $\operatorname{rank} \tilde{\bigwedge}^r d = \operatorname{rank} G - r$ and that $I(d) = I(\tilde{\bigwedge}^r d)$.

(2) Assume that the complex $\mathbb{F}_\bullet$ satisfies the rank conditions of Theorem 2.2 and depth $I(d_i) \geq i + k$ for $i = 1, \ldots, n$. One can use the previous remark to show that the module $M = H_0(\mathbb{F}_\bullet)$ is the k-th syzygy.

4. Linkage

Let R be a commutative local Gorenstein ring. Let I be a perfect ideal of codimension c. We have a finite free resolution

$$\mathbb{G}_\bullet : 0 \to G_c \xrightarrow{d_c} G_{c-1} \to \dots G_1 \xrightarrow{d_1} R \to R/I \to 0$$

Let $(x_1, \dots, x_c)$ be a regular sequence in I. We have a map of complexes

$$\alpha : K(x_1, \dots, x_c) \to \mathbb{G}_\bullet$$

extending the ring homomorphism

$$\alpha_0 : R/(x_1, \dots, x_c) \to R/I.$$

Theorem 4.1. *The dual of the mapping cone $C(\alpha)_\bullet$ is the free resolution of the colon ideal*

$$J := (x_1, \dots, x_c) : I.$$

Proof. We have an exact sequence

$$0 \to M \to R/(x_1, \dots, x_c) \to R/I \to 0$$

and the point is that the kernel $M = \mathrm{Ext}^c(R/J, R)$, so we really have an exact sequence

$$0 \to \mathrm{Ext}^c(R/J, R) \to R/(x_1, \dots, x_c) \to R/I \to 0$$

Similarly we have an exact sequence

$$0 \to \mathrm{Ext}^c(R/I, R) \to R/(x_1, \dots, x_c) \to R/J \to 0$$

which gives the statement of the theorem.

In order to see it, we pass to the ring $\bar{R} = R/(x_1, \dots, x_c)$. This is a complete intersection ring, so it is Gorenstein. In this ring we have two ideals $\bar{I} = I/(x_1, \dots, x_c)$, $\bar{J} = J/(x_1, \dots, x_c)$ and we see that $\bar{I} = 0 : \bar{J}$, $\bar{J} = 0 : \bar{I}$. This means that

$$\bar{I} = \mathrm{Hom}_{\bar{R}}(\bar{R}/\bar{J}, \bar{R}), \bar{J} = \mathrm{Hom}_{\bar{R}}(\bar{R}/\bar{I}, \bar{R})$$

This means we have exact sequences

$$0 \to \mathrm{Hom}_{\bar{R}}(\bar{R}/\bar{J}, \bar{R}) \to \bar{R} \to \bar{R}/\bar{I} \to 0$$

$$0 \to \mathrm{Hom}_{\bar{R}}(\bar{R}/\bar{I}, \bar{R}) \to \bar{R} \to \bar{R}/\bar{J} \to 0.$$

But the long exact sequences of Ext associated to regular sequence $(x_1, \dots, x_c)$ imply that

$$\mathrm{Hom}_{\bar{R}}(\bar{R}/\bar{J}, \bar{R}) = \mathrm{Ext}^c_R(R/J, R), \mathrm{Hom}_{\bar{R}}(\bar{R}/\bar{I}, \bar{R}) = \mathrm{Ext}^c_R(R/I, R).$$

This shows our claim. □

We have the following properties of linkage.

Proposition 4.2. *The relation of linkage satisfies the following*

(1) The ideal J is also perfect of codimension c.
(2) The relation is symmetric, i.e., $(x_1, \dots, x_c) : J = I$,

One says I is directly linked to J and writes $I \equiv J$.

We define the equivalence relation of linkage on the perfect ideals of codimension c in R to be the smallest equivalence relation containing the relation $I \equiv J$.

In low codimension the relation of linkage is very useful.

Proposition 4.3. *Let I be a perfect ideal of codimension 2. Then I is linked to a complete intersection.*

Proof. The finite free resolution of R/I is

$$0 \to R^{n-1} \xrightarrow{d_2} R^n \xrightarrow{d_1} R$$

where $d_1 = (x_1, \ldots, x_n)$, for some n. We can assume without loss of generality that (x_1, x_2) is a regular sequence in I (otherwise we change the basis in R^n). Then the mapping cone looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^{n-1} & \xrightarrow{d_2} & R^n & \xrightarrow{d_1} & R \\ & & \alpha_2 \uparrow & & \alpha_1 \uparrow & & \alpha_0 \uparrow \\ 0 & \longrightarrow & R & \xrightarrow{\beta_2} & R^2 & \xrightarrow{\beta_1} & R \end{array}$$

where β_2, β_1 are the differentials in the Koszul complex on x_1, x_2. Now, looking at the ranks of the maps modulo the maximal ideal $\mathfrak{m}$ in R we see that $\alpha_0 \otimes R/\mathfrak{m}$ has rank 1 and $\alpha_1 \otimes R/\mathfrak{m}$ has rank 2. This means that the minimal resolution of R/J has the format

$$0 \to R^{n-2} \xrightarrow{d_2'} R^{n-1} \xrightarrow{d_1'} R.$$

Continuing like that we arrive at a complete intersection. □

For perfect ideals of codimension three we can apply similar construction. Assume that we choose a regular sequence (x_1, x_2, x_3) in I such that x_1, x_2, x_3 are part of a minimal system of generators for I (i.e., their cosets are linearly independent modulo $\mathfrak{m}I$). A minimal resolution

$$0 \to R^m \xrightarrow{d_3} R^{m+n-1} \xrightarrow{d_2} R^n \xrightarrow{d_1} R$$

of R/I will then produce a minimal resolution

$$0 \to R^{n-3} \xrightarrow{d_3'} R^{m+n-1} \xrightarrow{d_2'} R^{m+3} \xrightarrow{d_1'} R$$

of the cyclic module R/J, where J is the linked ideal.

This means if we apply the procedure again (link by a regular sequence which is part of minimal generators of our ideal) we arrive at a resolution of format

$$0 \to R^m \xrightarrow{d_3''} R^{m+n-1} \xrightarrow{d_2''} R^n \xrightarrow{d_1''} R$$

again. Since at each stage we have many choices of regular sequences (x_1, x_2, x_3) we can produce many resolutions of the same format from a given one. This gives hope that resolutions of perfect ideals of codimension three occur in nice families, because from one of them we can produce nice families by linkage. Moreover, if for some reason we end up with the ideal I' such that, resolving ideal I', one of the Koszul relations of the ideal I' is among minimal syzygies, then there is additional cancellation, and we link I' to an ideal J with a smaller resolution of R/J. Such cases could be then handled by induction.

Notice that this method fails in codimension bigger than three, as the minimalization of the mapping cone will usually have much bigger ranks of modules.

5. Increasing depth: ideal transforms and geometric form of Acyclicity Criterion

We prove the geometric result on acyclicity of free complexes. It is based on homological algebra and it is essential for our approach.

Theorem 5.1. *Let $X = \operatorname{Spec} R$, and let $j : U \to X$ be an open immersion. Let*

$$\mathbb{G} : 0 \to G_n \to G_{n-1} \to \ldots \to G_1 \to G_0$$

be a complex of free R-modules (treated as a complex of sheaves over X) such that $\mathbb{G}|_U$ is acyclic. Then $H_n(\mathbb{G} \otimes j_* \mathcal{O}_U) = 0$, $H_{n-1}(\mathbb{G} \otimes j_* \mathcal{O}_U) = 0$, and the complex $\mathbb{G} \otimes j_* \mathcal{O}_U$ is acyclic if and only if $\mathcal{R}^i j_* \mathcal{O}_U = 0$ for $i = 1, \ldots, n-2$.

Proof. Before we start, let us decompose the complex $\mathbb{G}|_U$ into short exact sequences. Denoting $B_i = \Im(G_{i+1} \mid_U \to G_i \mid_U)$ we have exact sequences

$$0 \to B_i \to G_i \mid_U \to B_{i-1} \to 0$$

for $i = 2, \ldots, n$ (with $B_n = 0$) and

$$0 \to B_1 \to G_1 \mid_U \to G_0 \mid_U.$$

This induces long exact sequences

$$0 \to j_* B_i \to G_i \otimes j_* \mathcal{O}_U \to j_* B_{i-1} \to R^1 j_* B_i \ldots$$

as well as an exact sequence

$$0 \to j_* B_1 \to G_1 \otimes j_* \mathcal{O}_U \to G_0 \otimes j_* \mathcal{O}_U.$$

Next we show that vanishing of higher direct images implies acyclicity. Indeed, our vanishing implies that $R^i j_* B_{n-s} = 0$ for $1 \leq i \leq n-s-1$. So the long exact sequences above have last term zero and we get that $\mathbb{G} \otimes j_* \mathcal{O}_U$ is acyclic.

To prove the reverse implication let us proceed by induction on n. For $n = 2$ there is nothing to prove. For $n = 3$ we see from the exact sequences that $H_3(\mathbb{G} \otimes j_* \mathcal{O}_U) = H_2(\mathbb{G} \otimes j_* \mathcal{O}_U) = 0$ and $H_1(\mathbb{G} \otimes j_* \mathcal{O}_U) = \operatorname{Ker}(R^1 j_* G_3 \to R^1 j_* G_2)$. We need

Lemma 5.2. *Let M be an R-module. Let $\phi : F \to G$ be a map of free R-modules of finite rank. Denote by $I(\phi)$ the ideal of maximal minors of ϕ. Then $\phi \otimes M$ is a monomorphism if and only if $\operatorname{depth}_R(I(\phi), M) \geq 1$.*

Proof. This is a special case of Theorem 2, Appendix B from [22]. □

In our case $R^1 j_* \mathcal{O}_U$ is supported on $X \setminus U$ so $\operatorname{Ker}(R^1 j_* G_3 \to R^1 j_* G_2) = 0$ implies that $R^1 j_* \mathcal{O}_U = 0$, completing the case $n = 3$. Assume the result is proved for $n-1$ and the complex $j_* \mathbb{G} = \mathbb{G} \otimes j_* \mathcal{O}_U$ is acyclic. By induction (applied to $\mathbb{G}$ truncated at G_1) we have

$$R^1 j_* \mathcal{O}_U = R^2 j_* \mathcal{O}_U = \ldots = R^{n-3} j_* \mathcal{O}_U = 0.$$

Now our long exact sequences imply that $R^{n-3} j_* B_{n-2} = R^{n-4} j_* B_{n-3} = \ldots = R^1 j_* B_2$. We also have the exact sequence

$$0 \to R^{n-3} j_* B_{n-2} \to R^{n-2} j_* G_n \to R^{n-2} j_* G_{n-1}.$$

Also, from the exact sequences we can deduce that

$$H_1(\mathbb{G} \otimes j_* \mathcal{O}_U) = \mathrm{Ker}(R^1 j_* B_2 \to R^1 j_* G_2) = R^1 j_* B_2$$

This means that if $H_1(\mathbb{G} \otimes j_* \mathcal{O}_U) = 0$ then $R^1 j_* B_2 = 0$, so the map $R^{n-2} j_* G_n \to R^{n-2} j_* G_{n-1}$. is a monomorphism, which implies by Lemma 5.2 that $R^{n-2} j_* \mathcal{O}_U = 0$. □

6. Representation Theory.

In what follows we will use the representations of general linear groups and a little bit of representation theory of orthogonal Lie algebras. As the main reference we can use [31], chapter 2, however we will use slightly different notation.

A sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition of m if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ and $|\lambda| := \lambda_1 + \ldots + \lambda_s = m$. We identify the partitions $(\lambda_1, \ldots, \lambda_s)$ and $(\lambda_1, \ldots, \lambda_s, 0)$

Let F be an n-dimensional free module over a commutative ring R. We will denote by $S_{(\lambda_1,\ldots,\lambda_n)} F$ the Schur module corresponding to the weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Here $\lambda_1, \ldots, \lambda_n \in \mathbf{Z}$ and $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$. Again we define $|\lambda| := \lambda_1 + \ldots + \lambda_n$.

We have

$$S_{(\lambda_1+1,\lambda_2+1,\ldots,\lambda_n+1)} F = S_{(\lambda_1,\lambda_2,\ldots,\lambda_n)} F \otimes \bigwedge^n F,$$

and

$$S_{(\lambda_1,\lambda_2\ldots,\lambda_n)} F^* = S_{(-\lambda_n,-\lambda_{n-1},\ldots,-\lambda_1)} F.$$

If $\lambda_n \geq 0$ the Schur functor can be constructed from tensor powers of F by the usual construction with Young idempotent.

Since we mostly will deal with characteristic zero, let us now assume that the ring R is a $\mathbf{Q}$-algebra. The tensor product $S_\lambda F \otimes S_\mu F$ has a decomposition

$$S_\lambda F \otimes S_\mu F = \bigoplus_\nu S_\nu F^{\otimes c^\nu_{\lambda,\mu}}$$

where $c^\nu_{\lambda,\mu}$ are the Littlewood-Richardson coefficients. They are non-zero only if $|\lambda| + |\mu| = |\nu|$ and have a combinatorial interpretation given for example in [31], chapter 2.

We also have Cauchy formulas

$$S_t(F \otimes G) = \bigoplus_{|\lambda|=t,\lambda_n\geq 0} S_\lambda F \otimes S_\lambda G,$$

$$\bigwedge^t (F \otimes G) = \bigoplus_{|\lambda|=t,\lambda_n\geq 0} S_\lambda F \otimes S_{\lambda'} G,$$

where λ' is the conjugate partition.

Now we turn to representation theory of the orthogonal Lie algebra $\underline{so}(2n)$. Let V be an orthogonal space of dimension $2n$. The orthogonal form is denoted $\langle -, - \rangle$. We assume the form is given in the hyperbolic form, i.e., we have a basis $\{v_1, \ldots, v_n, \bar{v}_1, \ldots, \bar{v}_n\}$ of V such that $\langle v_i, \bar{v}_j \rangle = \delta_{i,j}$, $\langle v_i, v_j \rangle = 0$, $\langle \bar{v}_i, \bar{v}_j \rangle = 0$. The maximal toral subalgebra $\underline{h}$ is generated by elements h_i $(1 \leq i \leq n)$, where $h_i(v_i) = v_i$, $h_i(\bar{v}_i) = -\bar{v}_i$ and $h_i(v_j) = h_i(\bar{v}_j) = 0$ for $i \neq j$.

The irreducible representations $S_{[\lambda]} V$ correspond to highest weights $\lambda = \sum_{i=1}^n a_i \omega_i$ where $\omega_i = (1^i, 0^{n-i})$ for $1 \leq i \leq n-2$, $\omega_{n-1} = ((\frac{1}{2})^n)$, $\omega_n = ((\frac{1}{2})^{n-1}, -\frac{1}{2})$. We write $\lambda = (\lambda_1, \ldots, \lambda_n)$. We have two cases: if λ is a partition (i.e., $a_{n-1} + a_n$ is even) then $S_{[\lambda]} V$ can be constructed from the Schur functor $S_\lambda V$ as an $\underline{so}(2n)$-submodule generated by the canonical tableau (see [12], chapter 18).

The remaining representations have to be constructed using half-spinor representations. These are two representations $S_{[\omega_{n-1}]}V$ and $S_{[\omega_n]}V$ of dimension 2^{n-1} corresponding to highest weights ω_{n-1} and ω_n whose weights are $(\pm\frac{1}{2},\dots,\pm\frac{1}{2})$ with number of minus signs even for $S_{[\omega_{n-1}]}V$ and number of minus signs odd for $S_{[\omega_n]}V$.

If we write $V = H \oplus \bar{H}$ where H is a maximal isotropic space spanned by $v_1,\dots,v_n$, then

$$S_{[\omega_{n-1}]}V = \bigoplus_{j\ even} \bigwedge^j H, \qquad S_{[\omega_n]}V = \bigoplus_{j\ odd} \bigwedge^j H.$$

For more information, see [12], chapter 21.

7. Generic Rings.

We consider the free acyclic complexes $\mathbb{F}_\bullet$ (i.e complexes whose only nonzero homology group is $H_0(\mathbb{F}_\bullet)$) of the form

$$\mathbb{F}_\bullet :\ 0 \to F_n \to F_{n-1} \to \dots \to F_2 \to F_1 \to F_0$$

over commutative Noetherian rings R, with rank $F_i = f_i$ $(0 \le i \le n)$, $\mathrm{rank}(d_i) = r_i$ $(1 \le i \le n)$. The tuple $(f_0, f_1, \dots, f_{n-1}, f_n)$ is the *format* of the complex $\mathbb{F}_\bullet$. We always assume that $f_i = r_i + r_{i+1}$ $(0 \le i \le n)$.

For the resolutions of such format $(f_0, f_1, \dots, f_{n-1}, f_n)$ we say that a pair $(R_{gen}, \mathbb{F}_\bullet^{gen})$ where R_{gen} is a commutative ring and $\mathbb{F}_\bullet^{gen}$ is a free complex over R_{gen} is *a generic resolution* of this format if two conditions are satisfied:

1. The complex $\mathbb{F}_\bullet^{gen}$ is acyclic over R_{gen},
2. For every acyclic free complex $\mathbb{G}_\bullet$ of the same format over a Noetherian ring S there exists a ring homomorphism $\phi : R_{gen} \to S$ such that

$$\mathbb{G}_\bullet = \mathbb{F}_\bullet^{gen} \otimes_{R_{gen}} S.$$

Of particular interest is whether the ring R_{gen} is Noetherian, because it can be shown quite easily (see for example [4]) that a non-Noetherian (non-unique) generic pair always exists.

Theorem 7.1. *For every format $(f_0, f_1, \dots, f_n)$ a generic ring for resolutions of this format exists. It is, however, not unique, and in general non-Noetherian.*

Proof. We start with the generic complex of format $(f_0,\dots,f_n)$. Let R_0 be the coordinate ring of the variety of complexes (as described in the next section for $n = 2$) and let

$$\mathbb{F}_\bullet^{(0)} : 0 \to R_0^{f_n} \xrightarrow{d_n} R_0^{f_{n-1}} \to \dots \to R_0^{f_1} \xrightarrow{d_1} R_0^{f_0}$$

We define inductively a sequence of Noetherian rings R_m $(m \ge 0)$ and complexes $\mathbb{F}_\bullet^{(m)}$ of format $(f_0,\dots,f_n)$ over the rings R_m as follows. Assume that the pair $(R_m, \mathbb{F}_\bullet^{(m)})$ is 'constructed. Let $H_i^{(m)}$ denotes the i-th homology module of $\mathbb{F}_\bullet^{(m)}$. It is a finitely generated module, so let $\{q_1^{(m),i},\dots,q_{N(m,i)}^{(m),i}\}$ be a set of generators of the cycle submodule of $R_m^{f_i}$ for $i = 1,\dots,n$ (or just a set of cycles in $R_m^{f_i}$ whose image in $H_i^{(m)}$ generates it). We define the ring R_{m+1} as the R_m-algebra generated by the coordinates of elements $\{p_1^{(m),i}, \dots, p_{N(m,i)}^{(m),i}\}$ in $R_{m+1}^{f_{i+1}}$ such that $d_i(p_j^{(m),i}) = q_j^{(m),i}$ for all $i = 1, \dots, n$, $j = 1, \dots, N(m,i)$. This includes setting $\{q_1^{(m),n}, \dots, q_{N(m,n)}^{(m),n}\}$ equal to zero. We define the complex $\mathbb{F}_\bullet^{(m+1)} := \mathbb{F}_\bullet^{(m)} \otimes_{R_m} R_{m+1}$. We define the ring $R_{gen} = \lim_m R_m$ and $\mathbb{F}_\bullet^{gen} = \lim_m \mathbb{F}_\bullet^{(m)}$. We need to show that $H_i(\mathbb{F}_\bullet^{gen}) = 0$ for $i = 1, \dots, n$. Let us take a cycle z in $R_{gen}^{f_i}$. Each coordinate of z has finitely many terms, so all these terms occur already in some R_m. Thus z comes from a cycle in $R_m^{f_i}$. This means this cycle is a boundary over R_{m+1}, i.e., it is a boundary over R_{gen}. □

8. The case $n = 2$

In this section we produce an explicit generic ring for the resolutions of length two. This construction was first done by Hochster [15] with later improvements in [25], [26].

Let us fix a format (f_0, f_1, f_2) with the ranks (r_1, r_2), i.e., $f_2 = r_2, f_1 = r_1 + r_2$. Consider three free $\mathbf{Z}$ modules F_0, F_1, F_2 such that $\operatorname{rank} F_i = f_i$ $(i = 0, 1, 2)$.

We start with a generic complex, i.e., we take the independent variables $X_{j,i}$ and $Y_{k,j}$ $(1 \le i \le f_2, 1 \le j \le f_1, 1 \le k \le f_0)$. Consider two matrices $d_2 = (X_{j,i})$ and $d_1 = (Y_{k,j})$. Construct the ring

$$R_0 = \mathbf{Z}[\{X_{j,i}\}, \{Y_{k,j}\}]_{1 \le i \le f_2, 1 \le j \le f_1, 1 \le k \le f_0} / I(f_0, f_1, f_2)$$

where $I(f_0, f_1, f_2)$ is an ideal generated by relations

$$d_2 d_1 = 0, \bigwedge^{r_1+1} d_1 = 0.$$

Over the ring R_0 we have a "generic complex"

$$\mathbf{F}_\bullet^0 : F_2 \otimes_{\mathbf{Z}} R_0 \xrightarrow{d_2^0} F_1 \otimes_{\mathbf{Z}} R_0 \xrightarrow{d_1^0} F_0 \otimes_{\mathbf{Z}} R_0.$$

The differentials d_2^0 (resp., d_1^0) are the linear maps over R_0 given by matrices $(\bar{X}_{j,i}$, $(\bar{Y}_{k,j})$ over R_0 where $\bar{X}_{j,i}$ (resp., $\bar{Y}_{k,j}$) are the cosets of $X_{j,i}$ (resp., $Y_{k,j}$) in R_0.

Obviously this complex has a universality property with respect to all free complexes of format (f_0, f_1, f_2) over commutative rings.

However, it turns out that the complex $\mathbf{F}_\bullet^0$ is not acyclic over R_0. The reason is that this complex does not satisfy the first structure theorem of Buchsbaum-Eisenbud. Actually this theorem can be reinterpreted in terms of cycles in the first homology group of this complex.

In terms of depth it means that $\operatorname{depth}_{R_0} I_{r_2}(d_2^0) = 1$, $\operatorname{depth}_{R_0} I_{r_1}(d_1^0) = 1$. In order to increase the depth of $I_{r_2}(d_2^0)$ to two we can take the ideal transform of this ideal. Notice that the First Structure Theorem says that we have a factorization of $r_1 \times r_1$ minors of d_1:

$$M(K|J;Y) = \pm a_1(K) \cdot M(J' \,|[1, r_2]; X)$$

where I, J, K are subsets of cardinality r_1 and J' is the complement of J in the set $[1, r_1 + r_2]$, and $M(K|J;Y)$ (resp., $M(J'|[1, r_2]; X)$) are minors of Y (resp., X) on rows from K, columns from J (resp., rows from J', columns from $[1, r_2]$). Notice that this means that each of Buchsbaum-Eisenbud multipliers can be written as a factor of a minor of Y by an arbitrary maximal minor of X, so it is in the ideal transform of $I_{r_2}(d_2^0)$. Thus adding the Buchsbaum-Eisenbud multipliers to R_0 is necessary to increase the depth of $I_{r_2}(d_2)$ to 2.

It turns out that the ideal transform of $I_{r_2}(d_2^0)$ is exactly the ring R_a we would get from R_0 by adding to it all Buchsbaum-Eisenbud multipliers and dividing by all relations they would satisfy in all specializations. Over R_a we have a complex $\mathbf{F}_\bullet^a := \mathbf{F}_\bullet^0 \otimes_{R_0} R_a$ of format (f_0, f_1, f_2). This construction leads to the following.

Theorem 8.1. *The pair $(R_a, \mathbf{F}_\bullet^a)$ is a generic pair for resolutions of length two, of format (f_0, f_1, f_2). It has a universality property with respect to all acyclic free complexes of format (f_0, f_1, f_2) over commutative rings. The ring R_a has a filtration (which in characteristic zero is a direct sum decomposition)*

$$R_a = \bigoplus_{\boldsymbol{a},\boldsymbol{b},\alpha,\beta} S_{(\boldsymbol{a}-\boldsymbol{b}+\alpha_1,\ldots,\boldsymbol{a}-\boldsymbol{b}+\alpha_{r_2-1},\boldsymbol{a}-\boldsymbol{b})}F_2 \otimes$$
$$\otimes\, S_{(\boldsymbol{b}+\beta_1+\ldots,\boldsymbol{b}+\beta_{r_1-1},\boldsymbol{b},-\boldsymbol{a}+\boldsymbol{b},-\boldsymbol{a}+\boldsymbol{b}-\alpha_{r_2-1},\ldots,-\boldsymbol{a}+\boldsymbol{b}-\alpha_1)}F_1 \otimes S_{(0^{f_0-r_1},-\boldsymbol{b},-\boldsymbol{b}-\beta_{r_1-1},\ldots,-\boldsymbol{b}-\beta_1)}F_0.$$

Here we sum over all partitions α, β and natural numbers $\boldsymbol{a}, \boldsymbol{b}$. The entries of d_2 are a representation $F_2 \otimes F_1^*$ corresponding to $\alpha = (1)$, $\beta = \boldsymbol{a} = \boldsymbol{b} = 0$, the entries of d_1 are a representation $F_1 \otimes F_0^*$ corresponding to $\beta = (1)$, $\alpha = \boldsymbol{a} = \boldsymbol{b} = 0$, the entries of a_2 are a representation $\bigwedge^{r_2} F_2 \otimes \bigwedge^{r_2} F_1^*$ corresponding to $\alpha = \beta = \boldsymbol{b} = 0$, $\boldsymbol{a} = 1$, and the entries of a_1 are a representation $\bigwedge^{r_2} F_2^* \otimes \bigwedge^{r_1+r_2} F_1 \otimes \bigwedge^{r_1} F_0^*$ corresponding to $\alpha = \beta = \boldsymbol{a} = 0$, $\boldsymbol{b} = 1$.

The defining relations in the ring R_a can be made explicit. They involve usual straightening relations between the minors of differentials d_2, d_1, Plücker relations between the Buchsbaum-Eisenbud multipliers and additional relations between minors of d_2 (resp., d_1) and Buchsbaum-Eisenbud multipliers corresponding to Garnir relations on two columns, when one of the columns corresponds to Buchsbaum-Eisenbud multipliers. These relations are described in detail in [25], [26].

9. The case $n = 3$

Similarly to the case $n = 2$ the ring R_a can be constructed. It has a decomposition

$$R_a = \bigoplus_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\alpha,\beta,\gamma} S_{(\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_1,\ldots,\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_{r_3-1},\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c})}F_3 \otimes$$
$$\otimes S_{(\boldsymbol{b}-\boldsymbol{c}+\beta_1,\ldots,\boldsymbol{b}-\boldsymbol{c}+\beta_{r_2-1},\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_{r_2-1},\ldots,-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_1)}F_2$$
$$\otimes S_{(\boldsymbol{c}+\gamma_1,\ldots,\boldsymbol{c}+\gamma_{r_1-1},\boldsymbol{c},\boldsymbol{c}-\boldsymbol{b},\boldsymbol{c}-\boldsymbol{b}-\beta_{r_2-1},\ldots,\boldsymbol{c}-\boldsymbol{b}-\beta_1)}F_1$$
$$\otimes S_{(0^{f_0-r_1},-\boldsymbol{c},-\boldsymbol{c}-\gamma_{r_1-1},\ldots,-\boldsymbol{c}-\gamma_1)}F_0.$$

Again the defining relations (described in [25], [26]) involve usual straightening relations between the minors of differentials d_3, d_2, d_1, Plücker relations between the Buchsbaum-Eisenbud multipliers and additional relations between minors of d_3 (resp., d_2, d_1) and Buchsbaum-Eisenbud multipliers corresponding to Garnir relations on two columns, when one of the columns corresponds to Buchsbaum-Eisenbud multipliers.

Remark 9.1. The ring R_a has a similar description for any n (see [25], [26]). In general the complex $\mathbf{F}_\bullet^a$ that naturally exists over R_a is a universal complex acyclic in codimension one.

In the paper [32] for each format (f_0, f_1, f_2, f_3) the specific generic ring $\hat{R}_{gen}$ is constructed from R_a by a procedure of killing cycles. One can use the theory of representations of Kac-Moody Lie algebras to prove that this ring is indeed a generic ring. However the ring $\hat{R}_{gen}$ is Noetherian only in very few cases. The combinatorics of ranks works as follows. To three ranks (r_1, r_2, r_3) we associate the triple

$$(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1)$$

and we look at the graph $T_{p,q,r}$

$$\begin{array}{ccccccccccccc} x_{p-1} & - & x_{p-2} & \dots & x_1 & - & u & - & y_1 & \dots & y_{q-2} & - & y_{q-1} \\ & & & & & & | & & & & & & \\ & & & & & & z_1 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & \vdots & & & & & & \\ & & & & & & z_{r-2} & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & z_{r-1} & & & & & & \end{array}$$

The ring $\hat{R}_{gen}$ is Noetherian if and only if the graph $T_{p,q,r}$ is a Dynkin graph.

For the resolutions of cyclic modules (i.e., those with $r_1 = 1$, so $p = 2$) this means that $T_{p,q,r}$ has to be one of the following.

- Cases D_n $(n \geq 4)$, i.e., triples $(p, q, r) = (2, n-2, 2)$ and $(p, q, r) = (2, 2, n-2)$,
- Case E_6, i.e., a triple $(p, q, r) = (2, 3, 3)$,
- Cases E_7, i.e., triples $(p, q, r) = (2, 3, 4)$ and $(p, q, r) = (2, 4, 3)$,
- Cases E_8, i.e., triples $(p, q, r) = (2, 3, 5)$ and $(p, q, r) = (2, 5, 3)$.

Thus it is an important problem to describe the rings $\hat{R}_{gen}$ in these cases as explicitly as possible. Such description would allow us to "map" these resolutions as different generators could be considered to be coordinates in the variety of such resolutions.

The rings $\hat{R}_{gen}$ have remarkable properties. They have a decomposition analogous to R_a. If we write the decomposition for R_a given above in a more compact way,

$$R_a = \bigoplus_{\lambda \in \Lambda} S_{\alpha(\lambda)} F_3 \otimes S_{\beta(\lambda)} F_2 \otimes S_{\gamma(\lambda)} F_1 \otimes S_{\delta(\lambda)} F_0$$

then

$$\hat{R}_{gen} = \bigoplus_{\lambda \in \Lambda} S_{\beta(\lambda)} F_2 \otimes S_{\delta(\lambda)} F_0 \otimes V(\alpha(\lambda), \gamma(\lambda))$$

where $V(\theta)$ is a certain lowest weight module for the Kac-Moody Lie algebra corresponding to the diagram $T_{p,q,r}$. Here θ is the dominant integral weight for $T_{p,q,r}$, i.e., labeling the nodes of $T_{p,q,r}$ by integers.

The connection of the graph $T_{p,q,r}$ with our free resolution is as follows. After removing the vertex z_1 we get the graph $T_{p,q,r}$ with two connected components. The component containing vertices x_i, u, y_j $(1 \leq i \leq p-1, 1 \leq j \leq q-1)$ is thought of as the Dynkin diagram of the root system of F_1 and the component containing vertices z_k $(2 \leq k \leq r-1)$ is thought of as the Dynkin diagram of the root system of F_3. The nice thing that happens is that $\hat{R}_{gen}$ has a bigger symmetry expressed by the action of the Lie algebra corresponding to the graph $T_{p,q,r}$.

Besides studying the structure of the generic ring $\hat{R}_{gen}$ it is important to study the examples of these resolutions occurring in algebraic geometry in order to understand how their structure might help in understanding these examples.

In these notes we will not use representation theory of exceptional Lie algebras or Kac-Moody Lie algebras so we will look mostly at the D_n cases.

One of the main conjectures ([11]) we made whose aim would be to show that Dynkin formats are indeed special is the following

Conjecture 9.1. *The Dynkin formats are precisely the formats such that any perfect ideal I in a regular local ring R such that the minimal free resolution of R/I has a Dynkin format is in the linkage class of a complete intersection.*

We will refer to this conjecture as the LICCI Conjecture. One can consult [11] for the examples based on Macaulay inverse systems showing that for non-Dynkin format we cannot hope it satisfies the condition of LICCI conjecture.

10. Example: formats $(1, n, n, 1)$

Let us start with the simplest formats $(1, n, n, 1)$.

In this format we deal with the special orthogonal Lie algebra $\underline{so}(U)$ where U is the orthogonal space

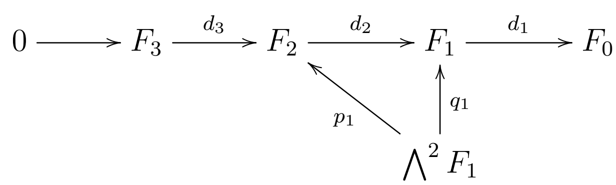
$$U = F_1^* \oplus F_1$$

with the quadratic form which is a duality pairing on $F_1^* \oplus F_1$. The grading on this Lie algebra is

$$\underline{so}(U) = \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1$$

where $\underline{g}_0 = \underline{sl}(F_1) \oplus \mathbf{C} = \underline{gl}(F_1)$, $\underline{g}_1 = \bigwedge^2 F_1$, $\underline{g}_{-1} = \underline{g}_1^*$.

The generic ring is obtained from the ring R_a in one stage, by lifting the cycle giving Koszul relations



which gives the defect $F_3^* \otimes \bigwedge^2 F_1$.

It was proved already in [25] that in this way we get a generic ring. This was done by producing a family of complexes over $Sym(\bigwedge^2 F_1)$ resolving certain family of modules, without noticing that these are parabolic BGG resolutions. One has to mention that at that time the parabolic BGG resolutions were not yet invented.

Let us describe the key representations in $\hat{R}_{gen}$. They are

$$W(d_3) = F_2^* \otimes [\bigoplus_{k\geq 0} S_{1-k}F_3 \otimes \bigwedge^{2k} F_1],$$

$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes F_1],$$

$$W(a_2) = \mathbf{C} \otimes [\bigoplus_{k\geq 0} S_k F_3^* \otimes \bigwedge^{k+1} F_1].$$

These representations give generators of the generic ring $\hat{R}_{gen}$. Let us make a few basic observations. Three representations described above acquire the grading induced by the grading on $\underline{so}(U)$. In the lowest degree we get just the representations d_3, d_2 and a_2 from R_a. This is a general phenomenon: for each irreducible representation in $\hat{R}_{gen}$ its lowest degree term will just give the corresponding representation from R_a. Looking at the next degree term in our three representations we see the tensors $F_2^* \otimes \bigwedge^2 F_1$, $F_2 \otimes F_3^* \otimes F_1$ and $F_3^* \otimes \bigwedge^3 F_1$. These tensors can be thought of as maps $\bigwedge^2 F_1 \to F_2$, $F_1 \otimes F_2 \to F_3$ and $\bigwedge^3 F_1 \to F_3$. These are the components of the multiplicative structure on $\mathbf{F}_\bullet$. Why do we know that? It was already proved by Buchsbaum and Eisenbud that every finite free resolution of length three has an associative, graded commutative algebra structure. So the components of this structure have to sit in the generic ring. But one can easily see that these three representations occur in $\hat{R}_{gen}$ only once, so they have to be the tensors giving the multiplicative structure.

The other components of three representations $W(d_3)$, $W(d_2)$, $W(a_2)$ also have similar interpretation in terms of the resolution. Let us look at some resolution

$$0 \to G_3 \xrightarrow{d_3} G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} R$$

of format $(1, n, n, 1)$ over some Noetherian commutative ring R. We have a comparison map from the Koszul complex on d_1 to our resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_3 & \xrightarrow{d_3} & G_2 & \xrightarrow{d_2} & G_1 & \xrightarrow{d_1} & R \\ & & \uparrow{\scriptstyle \alpha_3} & & \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_0} \\ \bigwedge^4 G_1 & \xrightarrow{\delta_4} & \bigwedge^3 G_1 & \xrightarrow{\delta_3} & \bigwedge^2 G_1 & \xrightarrow{\delta_2} & G_1 & \xrightarrow{d_1} & R \end{array}$$

The maps α_2, α_3 give us the first graded components of $W(d_3)$ and $W(a_2)$.

Now, the composition $d_3\alpha_3\delta_4 = 0$, but d_3 is injective, so $\alpha_3\delta_4 = 0$. But the entries of the matrix δ_4 involves only the four generators of the resolved ideal, so the last equation can be rewritten as such relation. This can be converted to the claim that we have a complex

$$\bigwedge^5 G_1 \xrightarrow{\delta_5} \bigwedge^4 G_1 \xrightarrow{\alpha_3} G_1 \xrightarrow{d_1} R,$$

where by abuse of notation we denote by α_3 the contraction by α_3.

So we have a comparison map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_3 & \xrightarrow{d_3} & G_2 & \xrightarrow{d_2} & G_1 & \xrightarrow{d_1} & R \\ & & \uparrow{\scriptstyle \beta_3} & & \uparrow{\scriptstyle \beta_2} & & \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_0} \\ \bigwedge^6 G_1 & \xrightarrow{\delta_6} & \bigwedge^5 G_1 & \xrightarrow{\alpha_3} & \bigwedge^2 G_1 & \xrightarrow{\delta_2} & G_1 & \xrightarrow{d_1} & R \end{array}$$

The maps β_2, β_3 give us the second graded components of of $W(d_3)$ and $W(a_2)$. Now, the composition $d_3\beta_3\delta_6 = 0$, but d_3 is injective, so $\beta_3\delta_6 = 0$. But the entries of the matrix δ_6 involves only the four generators of the resolved ideal, so the last equation can be rewritten as such relation. This can be converted to the claim that we have a complex

$$\bigwedge^7 G_1 \xrightarrow{\delta_7} \bigwedge^6 G_1 \xrightarrow{\beta_3} G_1 \xrightarrow{d_1} R,$$

where by abuse of notation we denote by β_3 the contraction by β_3, and we continue like that to get all higher components of $W(d_3)$ and $W(a_2)$.

Again we know that the representations giving higher components of $W(d_3)$ and $W(a_2)$ have to specialize to the factorizations we just constructed because they are the only representations of this type occurring in $\hat{R}_{gen}$.

For other Dynkin formats it would be desirable to construct similar interpretations.

Now we turn to another problem, we want to look at the perfect ideals of our format. One of the ways to get them is to look at the open set

$$U_{CM} \subset \mathrm{Spec}(\hat{R}_{gen})$$

consisting of points where the complex $(\mathbf{F}^{gen}_\bullet)^*$ is acyclic.

The format $(1, n, n, 1)$ is helpful because we know the answer by Buchsbaum-Eisenbud Theorem: perfect ideals with a resolution of format $(1, n, n, 1)$ exist if and only if n is odd and they are given by Pfaffians of an odd-sized skew-symmetric matrix. So let us look at the format $(1, n, n, 1)$ with n odd. One can recall that the key point in Buchsbaum-Eisenbud Theorem is that for a resolution of a perfect ideal of format $(1, n, n, 1)$ the multiplication

$$F_1 \otimes F_2 \to F_3 = R$$

gives a perfect pairing of F_1 and F_2. So this means that the entries of that tensor do not lie in a maximal ideal of R (assuming our resolution is over a local ring R).

But this multiplication is the tensor in the top graded components of $W(d_2)$.

This gives an idea of looking at the top graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$.

Let us look at the "forbidden" format $(1, n, n, 1)$, n even. In this case we see that the three top components of our representations are: $F_2^* \otimes F_3^*$, $F_2 \otimes F_1$ and F_1^*. These three tensors can be thought of as three maps

$$F_3^* \to F_2 \to F_1^* \to R$$

over $R = \hat{R}_{gen}$. Moreover, it is easily seen that these tensors give us a complex, as there are no representations in $\hat{R}_{gen}$ giving compositions of these maps.

This also brings out the key difference between the formats $(1, n, n, 1)$ for n even and odd. For n even two half-spinor representations are self dual, for n odd they are dual to each other. It is natural to conjecture the following

Conjecture 10.1. *The point from* $\mathrm{Spec}(\hat{R}_{gen})$ *is in* U_{CM} *if the tensors giving top graded components of* $W(d_3)$, $W(d_2)$, $W(a_2)$ *give a split exact complex.*

Let us denote U_{split} the open set of points in $\mathrm{Spec}(\hat{R}_{gen})$ where the tensors giving top graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$ give a split exact complex.

This gives the following idea. Since the ring $\hat{R}_{gen}$ has an action of our orthogonal Lie algebra, in order to see a generic point of U_{split} we can use the involution of $\underline{so}(U)$ interchanging highest and lowest weights, and then it is enough to calculate all the factorizations giving components of $W(d_3)$, $W(d_2)$, $W(a_2)$ using the defect variables. We should get a nice resolution of a perfect ideal.

Let us see what happens for the smallest format $(1,4,4,1)$. We take the split exact complex

$$0 \to R \xrightarrow{d_3} R^4 \xrightarrow{d_2} R^4 \xrightarrow{d_1} R.$$

Let $\{e_1, e_2, e_3, e_4\}$ will be the basis of F_1, $\{f_1, f_2, f_3, f_4\}$ will be the basis of F_2, $\{g\}$ will be the basis of F_3, such that the differentials in these bases are given by matrices

$$d_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, d_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, d_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix},$$

Calculating the multiplicative structure we get

$$e_i \cdot e_j = b_{ij} f_4,$$

for $1 \leq i, j \leq 3$,

$$e_i \cdot e_4 = -f_i + b_{i4} f_4,$$

for $1 \leq i \leq 3$.

Let us calculate the tensor giving the component $v_2^{(3)}$. This is a factorization

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^2 F_3 & \xrightarrow{d_3} & F_3 \otimes F_2 & \xrightarrow{d_3} & S_2 F_2 & \xrightarrow{S_2(d_2)} & S_2 F_1 \\ & & & & & \nwarrow^{v_2^{(3)}} & \uparrow{\scriptstyle q_2} & & \\ & & & & & & \bigwedge^4 F_1 & & \end{array}$$

with $q_2 = S_2(p_1)$. We get that

$$v_2^{(3)}(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = b_{12} g \otimes (e_3 \cdot e_4) - b_{13} g \otimes (e_2 \cdot e_4) + b_{14} g \otimes (e_1 \cdot e_4)$$

which, written as a matrix, gives

$$v_2^{(3)} = \begin{pmatrix} -b_{23} \\ b_{13} \\ -b_{12} \\ \mathrm{Pf}((b_{ij})) \end{pmatrix}.$$

Note that after row operations we just get the matrix

$$\begin{pmatrix} -b_{23} \\ b_{13} \\ -b_{12} \\ 0 \end{pmatrix}.$$

Lifting similarly the other maps we get that (after applying the row and column operations)

$$v_1^{(2)} = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 \\ -b_{12} & 0 & b_{23} & 0 \\ -b_{13} & -b_{23} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

ans

$$v_2^{(1)} = \begin{pmatrix} b_{23} & b_{13} & b_{12} & 0 \end{pmatrix}.$$

This means that the complex $\mathbf{F}_\bullet^{top}$ is a generic resolution of format $(1,3,3,1)$ plus a splitting factor $R\xrightarrow{1}R$. Similarly (it is the easiest to lift $v_1^{(2)}$) we see that for the format $(1,n,n,1)$, n even we get in the same way that the complex $\mathbf{F}_\bullet^{top}$ is a direct sum of generic resolution of a perfect ideal of format $(1,n-1,n-1,1)$ and a splitting factor $R\xrightarrow{1}R$.

11. Example: formats $(1,4,n,n-3)$

In this format we deal with the special orthogonal Lie algebra which can be identified with the orthogonal space

$$U = F_3^*\oplus\bigwedge^2 F_1\oplus F_3$$

with the quadratic form which is a duality pairing on $F_3^*\oplus F_3$ and the exterior multiplication on $\bigwedge^2 F_1$. The grading on this Lie algebra is

$$\underline{so}(U)=\underline{g}_{-2}\oplus\underline{g}_{-1}\oplus\underline{g}_0\oplus\underline{g}_1\oplus\underline{g}_2$$

where $\underline{g}_0=\underline{sl}(F_3)\oplus\underline{sl}(F_1)\oplus\mathbf{C}$, $\underline{g}_1=F_3^*\otimes\bigwedge^2 F_1$, $\underline{g}_2=\bigwedge^2 F_3^*\otimes\bigwedge^4 F_1$, $\underline{g}_{-i}=\underline{g}_i^*$.

The generic ring is obtained from the ring R_a in two stages. First, we lift the cycle giving Koszul relations

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_3 & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0\\ & & & & & \nwarrow^{p_1} & \uparrow{\scriptstyle q_1} & & \\ & & & & & & \bigwedge^2 F_1 & & \end{array}$$

which gives the defect $F_3^*\otimes\bigwedge^2 F_1$ and then we kill the cycle

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^2 F_3 & \xrightarrow{d_3} & F_3\otimes F_2 & \xrightarrow{d_3} & S_2F_2 & \xrightarrow{S_2(d_2)} & S_2F_1\\ & & & & & \nwarrow^{p_2} & \uparrow{\scriptstyle q_2} & & \\ & & & & & & \bigwedge^4 F_1 & & \end{array}$$

where $q_2=S_2(p_1)$, with defect $\bigwedge^2 F_3^*\otimes\bigwedge^4 F_1$.

Three key representations in $\hat{R}_{gen}$ have the following decompositions.

$$W(d_3)=F_2^*\otimes[F_3\oplus\bigwedge^2 F_1\oplus F_3^*\otimes\bigwedge^4 F_1],$$

$$W(d_2)=F_2\otimes[F_1^*\otimes\bigwedge^{even} F_3^*\oplus F_1\otimes\bigwedge^{odd} F_3^*],$$

$$W(a_2)=\mathbf{C}\otimes[F_1\otimes\bigwedge^{even} F_3^*\oplus F_1^*\otimes\bigwedge^{odd} F_3^*].$$

Notice the three top components of these three representations. In $\dim F_3=2m$ is even they are: $F_2^*\otimes F_3^*$, $F_2\otimes F_1^*$ and F_1. These three can be thought of as three maps

$$F_3^*\otimes\hat{R}_{gen}\xrightarrow{d_3^{top}}F_2\otimes\hat{R}_{gen}\xrightarrow{d_2^{top}}F_1\otimes\hat{R}_{gen}\xrightarrow{d_1^{top}}\hat{R}_{gen}.$$

It is not difficult to see they form a complex over $\hat{R}_{gen}$. Similarly, if $\dim F_3 = 2m+1$ is odd, the three top graded components of our representations are: $F_2^* \otimes F_3^*$, $F_2 \otimes F_1$ and F_1^*. These three can be thought of as three maps

$$F_3^* \otimes \hat{R}_{gen} \xrightarrow{d_3^{top}} F_2 \otimes \hat{R}_{gen} \xrightarrow{d_2^{top}} F_1^* \otimes \hat{R}_{gen} \xrightarrow{d_1^{top}} \hat{R}_{gen}.$$

It is not difficult to see they form a complex over $\hat{R}_{gen}$.

The important conjecture announced in [10] is that the open set U_{CM} of points in Spec $\hat{R}_{gen}$ is equal to the open set U_{split} where the localization of the complex $\mathbf{F}_\bullet^{gen*}$ is acyclic. The conjecture is that these are the points where the complex $\mathbf{F}_\bullet^{top}$ is split exact. This means that if we want to see the general point in U_{CM} as a resolution of a perfect ideal, we can do the reverse calculation. We can set the original complex $\mathbf{F}_\bullet$ to be split exact and then, working over polynomial ring in defect variables, we can calculate the differentials d_i^{top} for this complex. If $U_{split} \subset U_{CM}$, then we should get a resolution of a perfect ideal.

Let us do this calculation for this format. We will calculate d_3^{top} for the split complex $\mathbf{F}_\bullet$. Let us start with the complex

$$R^{n-3} \xrightarrow{d_3} R^n \xrightarrow{d_2} R^4 \xrightarrow{d_1} R,$$

where

$$d_3 = \begin{pmatrix} 0_{3\times n} \\ I_{n-3} \end{pmatrix}$$

$$d_2 = \begin{pmatrix} I_3 & 0_{3\times n} \\ 0_{1\times 3} & 0_{1\times(n-3)} \end{pmatrix},$$

$$d_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here I_r denotes an $r\times r$ identity matrix and $0_{a\times b}$ is an $a\times b$ zero matrix.

We denote $\{e_1, e_2, e_3, e_4\}$ (resp., $\{f_1 \ldots, f_n\}$, $\{g_1, \ldots, g_{n-3}\}$) the basis in $F_1 \otimes R$ (resp., $F_2 \otimes R$, $F_3 \otimes R$). We will calculate d_3^{top}. We start with the multiplicative structure. We get

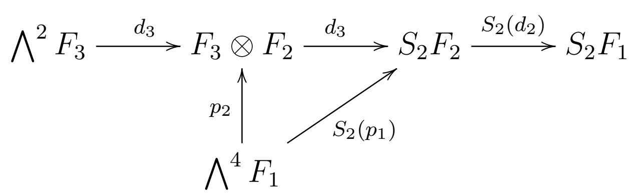
$$e_i \cdot e_j = \sum_{s=1}^{n-3} b_{ij;s} f_{s+3}$$

for $1 \le i, j \le 3$,

$$e_i \cdot e_4 = -f_i + \sum_{s=1}^{n-3} b_{i4;s} f_{s+3},$$

for $1 \le i \le 3$.

To calculate the top component we need to remember that this is a map which lifts the following cycle



We have

$$S_2(p_1)(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = (e_1 \cdot e_2)(e_3 \cdot e_4) - (e_1 \cdot e_3)(e_2 \cdot e_4) + (e_2 \cdot e_3)(e_1 \cdot e_4).$$

So we have

$$p_2(e_1\wedge e_2\wedge e_3\wedge e_4) = (\sum_{s-=1}^{n-3} b_{12;s}g_s)\otimes(-f_3+\sum_{s=1}^{n-3} b_{34;s}f_s) - (\sum_{s-=1}^{n-3} b_{13;s}g_s)\otimes(-f_2+\sum_{s=1}^{n-3} b_{24;s}f_s)$$

$$+(\sum_{s-=1}^{n-3} b_{23;s}g_s)\otimes(-f_1+\sum_{s=1}^{n-3} b_{14;s}f_s) + \sum_{1\le s<t\le n-3} c_{s,t}(g_s\otimes f_{t+3} - g_t\otimes f_{s+3}).$$

Writing this tensor as an $n\times(n-3)$ matrix we get

$$\begin{pmatrix} -b_{23;1} & -b_{23;2} & \dots \\ +b_{13;1} & +b_{13;2} & \dots \\ -b_{12;1} & -b_{12;2} & \dots \\ c_{1,1}+b_{12;1}b_{34;4}-b_{13;1}b_{24;4}+b_{23;1}b_{14;4} & c_{2,1}+b_{12;2}b_{34;4}-b_{13;2}b_{24;4}+b_{23;2}b_{14;4} & \dots \\ \dots & \dots & \dots \\ c_{1,n-3}+b_{12;1}b_{34;n-3}-b_{13;1}b_{24;n-3}+b_{23;1}b_{14;n-3} & c_{2,n-3}+b_{12;2}b_{34;n-3}-b_{13;2}b_{24;n-3}+b_{23;2}b_{14;n-3} & \dots \end{pmatrix}.$$

$$\begin{pmatrix} \dots & -b_{23;n-3} \\ \dots & +b_{13;n-3} \\ \dots & -b_{12;n-3} \\ \dots & c_{n-3,1}+b_{12;n-3}b_{34;4}-b_{13;n-3}b_{24;4}+b_{23;n-3}b_{14;4} \\ \dots & \dots \\ \dots & c_{n-3,n-3}+b_{12;n-3}b_{34;n-3}-b_{13;n-3}b_{24;n-3}+b_{23;n-3}b_{14;n-3} \end{pmatrix}.$$

Adding to the s-th row W_s the combination $b_{34;s}W_1 - b_{24;s}W_2 + b_{14;s}W_3$ gives us a matrix

$$\begin{pmatrix} -b_{23;1} & -b_{23;2} & \dots & -b_{23;n-3} \\ +b_{13;1} & +b_{13;2} & \dots & +b_{13;n-3} \\ -b_{12;1} & -b_{12;2} & \dots & -b_{12;n-3} \\ c_{1,1} & c_{2,1} & \dots & c_{n-3,1} \\ \dots & \dots & \dots & \dots \\ c_{1,n-3} & c_{2,n-3} & \dots & c_{n-3,n-3} \end{pmatrix}.$$

Notice that the variables $b_{i4;s}$ disappear. They are exactly the variables which, written in the terms of roots of D_n, have label 0 on the vertex n. This is a general phenomenon which happens for other Dynkin formats.

12. Example: formats $(1,5,6,2)$

In this example the grading on algebra $\underline{g}(E_6)$ is as follows.

$$\underline{g}(E_6) = \underline{g}(E_6)_{-2}\oplus\underline{g}(E_6)_{-1}\oplus\underline{g}(E_6)_0\oplus\underline{g}(E_6)_1\oplus\underline{g}(E_6)_2,$$

where $\underline{g}(E_6)_0 = \underline{sl}(F_3)\times\underline{sl}(F_1)\times\mathbf{C}$ and

$$\underline{g}(E_6)_1 = F_3^*\otimes\bigwedge^2 F_1, \underline{g}(E_6)_2 = \bigwedge^2 F_3^*\otimes\bigwedge^4 F_1.$$

We get the generic ring from R_a in two steps. First we kill a cycle q_1 given by Koszul relations

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_3 & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 \\ & & & & & {\scriptstyle p_1}\nwarrow & \uparrow{\scriptstyle q_1} & & \\ & & & & & & \bigwedge^2 F_1 & & \end{array}$$

This gives the defect $F_3^* \otimes \bigwedge^2 F_1$. Then we kill the cycle

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^2 F_3 & \xrightarrow{d_3} & F_3 \otimes F_2 & \xrightarrow{d_3} & S_2F_2 & \xrightarrow{S_2(d_2)} & S_2F_1 \\ & & & & & {\scriptstyle p_2}\nwarrow & \uparrow{\scriptstyle q_2} & & \\ & & & & & & \bigwedge^4 F_1 & & \end{array}$$

where $q_2 = S_2(p_1)$, with defect $\bigwedge^2 F_3^* \otimes \bigwedge^4 F_1$.

At each stage we then divide by annihilators of the ideals $I(d_3)$, $I(d_2)$, $I(d_1)$ and take ideal transforms of the ideals $I(d_3)$, $I(d_2)$. Three critical representations have the form

$$W(d_3) = F_2^* \otimes [F_3 \oplus \bigwedge^2 F_1 \oplus F_3^* \otimes \bigwedge^4 F_1 \oplus \bigwedge^2 F_3^* \otimes S_{2,1,1,1,1}F_1],$$

$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes F_1 \oplus \bigwedge^2 F_3^* \otimes \bigwedge^3 F_1 \oplus S_{2,1}F_3^* \otimes \bigwedge^5 F_1],$$

$$W(d_1) = \mathbf{C} \otimes [F_1 \oplus F_3^* \otimes \bigwedge^3 F_1 \oplus$$

$$\oplus [\bigwedge^2 F_3^* \otimes \bigwedge^4 F_1 \otimes F_1 \oplus S_2F_3^* \otimes \bigwedge^5 F_1] \oplus S_{2,1}F_3^* \otimes S_{2,2,1,1,1}F_1 \oplus S_{2,2}F_3^* \otimes S_{2,2,2,2,1}F_1]$$

Note that (and this happens for arbitrary format different than $(1,n,n,1)$) the lowest weight components are d_3, d_2, a_2 respectively and the next graded components of these three representations are the tensors giving the multiplicative structure on resolution $\mathbf{F}_\bullet$.

13. Examples from algebra and geometry

13.1. **Artin algebras and Macaulay inverse systems.** We work with the polynomial ring $S := K[x,y,z]$. It is a graded ring $S = \oplus_{i\ge 0} S_i$. We will consider the homogeneous ideals I such that S/I is an Artin ring. This means certain power of the irrelevant ideal $\mathfrak{m} = \oplus_{i>0} S_i$ is contained in I. We are interested in minimal graded free resolution of S/I over S. It will have length 3. In particular we would like to know when such resolution has Dynkin format.

We define the Hilbert function $h_I(t)$ to be the polynomial

$$h_I(t) = \sum_{i\ge 0} (\dim(S/I)_i) t^i.$$

Macaulay inverse systems allow to produce interesting ideals I. We start with the dual variables $x' = \partial/\partial x$, $y' = \partial/\partial y$, $z' = \partial/\partial z$. To any subspace $V \subset T := K[x', y', z']$ we associate its orthogonal complement ideal

$$I(V) = \{f \in K[x, y, z] \mid f(V) = 0\}$$

where f acts on T via differential operators.

The basic example is the case of $\dim V = 1$. Then the ideal $I(V)$ is Gorenstein, and one proves that any homogeneous Gorenstein ideal $I \subset S$ arrives in that way.

We are interested in families of subspaces V (where we fix dimension of V in each degree) such that for general choice of element from V the cyclic module $S/I(V)$ has a resolution of Dynkin format. Alternatively, we can look at the families of ideals I with fixed Hilbert function of S/I and try to decide when such family could have resolutions of Dynkin format.

For this one does something called Sample Calculation which is best explained by example:

Example 13.1. Let us consider the ideals I such that the Hilbert function of S/I is $(1, 3, 6, 6, 2)$. What is the expected resolution of S/I? It is gotten by polynomial

$$(1 + 3t + 6t^2 + 6t^3 + 2t^4)(1 - 3t + 3t^2 - t^3) = 1 - 4t^3 - t^4 + 5t^5 - 2t^7$$

which means we expect ideals with 4 cubic and one quartic generator, with 5 relations in degree 5 and 2 second syzygies in degree 7.

The point is trying to verify the LICCI Conjecture in such cases.

Here are some examples of Hilbert functions of S/I that were produced by Sema Güntürkün:

$(1, 3, 4, 4)$, type E_8, $(1, 3, 5, 3)$, type E_7, $(1, 3, 6, 4, 2)$, type E_7, $(1, 3, 6, 5, 3)$, type E_7. We would like to understand how to produce concrete examples of such ideals by Macaulay 2, and then how to test how they link. Here is the procedure that should in principle work. Let us look at the example.

Example 13.2. Let us look again at the ideals I such that S/I has the Hilbert function $(1, 3, 6, 6, 2)$. In this case it is easy to produce such examples by taking the subspace V of dimension 2 in S_4 and taking the ideal $I(V)$. The resolution one gets is

$$0 \to S^2(-7) \to S^5(-5) \to S(-4) \oplus S^4(-3) \to S$$

Now in general we have a regular sequence of three elements of degree 3 in I. For a linked ideal J the resolution of S/J will be

$$0 \to S(-5) \oplus S(-6) \to S^5(-4) \to S^2(-2) \oplus S(-3) \to S.$$

But then, in general, if we can find a regular sequence of degrees $2, 2, 3$ in J then we expect one Koszul relation to be among minimal syzygies. This should link to a smaller resolution, so this ideal should be licci.

Such method in principle works numerically, but we are not sure we can always find required regular sequence in low degrees and that the maximal cancellation of ranks in the resulting mapping cone occurs.

Another interesting aspect is trying to find the irreducible components of Hilbert schemes $\mathrm{Hilb}(S, h(t))$ of varieties of homogeneous ideals I such that the Hilbert function $h_I(t)$ is $h(t)$.

There is another phenomenon regarding finite free resolutions that is interesting in this context.

Example 13.3. Take $h(t) = 1 + 3t + t^2 + t^3 + t^4 + t^5$. The Sample Calculation gives

$$(1 + 3t + t^2 + t^3 + t^4 + t^5)(1 - 3t + 3t^2 - t^3) = 1 - 5t^2 + 6t^3 - 2t^4 - t^6 + 2t^7 - t^8.$$

So the expected resolution splits into two parts: the resolution of type E_6, resolving the general ideal J with Hilbert function $1 + 3t + \sum_{i \geq 2} t^i$ (the algebra S/J is not Artinian, it is a point) and Koszul complex in two variables given by killing the element in J in degree 6. This phenomenon happens for some Hilbert functions and it deserves to be investigated. It could produce resolutions of Dynkin formats resolving non-perfect ideals.

13.2. **Points in $\mathbf{P}^3$.** The ACM sets of points in $\mathbf{P}^3$ also give the length three resolutions. The set of Hilbert functions one could obtain is contained in the Hilbert functions of homogeneous Artinian factors of $K[X, Y, Z]$, but it contains the Hilbert functions of ACM curves of degree d of genus g in $\mathbf{P}^4$ (see below).

13.3. **Curves in $\mathbf{P}^3$.** Non-Cohen-Macaulay curves in $\mathbf{P}^3$ also could lead to resolutions of length three. Giuffrida and Maggioni [13] studied curves lying on a smooth cubic surface in $\mathbf{P}^3$. Exhibiting such examples by Macaulay 2 could also be interesting.

13.4. **ACM curves in $\mathbf{P}^4$.** Caroline and Laurent Gruson in [14] calculated possible Hilbert functions and resolutions of curves of genus g and degree d, up to $d = 15$. There are several cases of Dynkin types, but they seem always to be LICCI. It would be good to confirm it at least for a general curve in such cases.

The smallest cases are: $d = 9, g = 6$ (type E_6), $d = 11, g = 9$ (type E_7), $d = 12, g = 11$ (type E_7), $d = 13, g = 15$ (type E_6).

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