Project Simplex

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Abstract

The aim of the project is to visualize the so called *Bruhat-Tits Building* of the group $\mathrm{GL}_3(\mathbb{Q}_2)$, with the intent of generalizing the methods for use with other building types.

Drawing the building is a three step process. Using both the Bruhat decompostion and the root group decomposition of this group, we obtain methods for generating all components (the chambers), deciding which are adjacent (generating the chamber graph) and finally positioning them in space.

Coxeter complexes

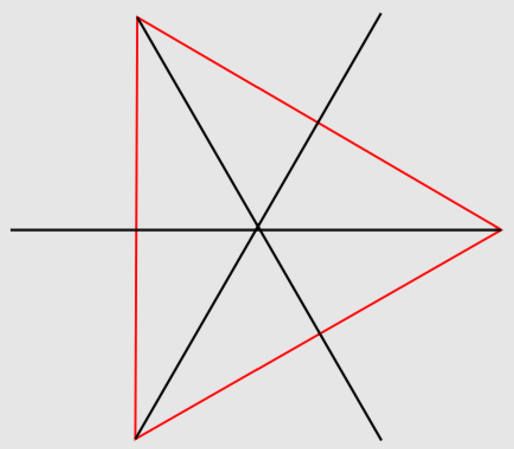

Figure 1: The three reflections generating all linear isometries of the equilateral triangle. This gives the Coxeter complex for S_3.

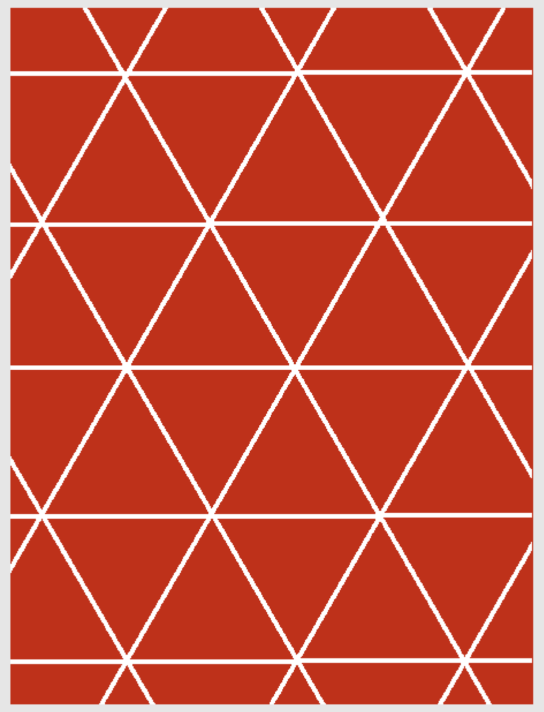

Figure 2: Part of the Coxeter complex of $\mathbb{Z}^2 \rtimes S_3$.

Definition (Coxeter system[1])

A *Coxeter system* is a group W together with a set of generators S, called the set of *simple reflections*, such that W allows a presentation:

$$W = \langle S \mid \forall i,j : s_i^2 = (s_i s_j)^{m(i,j)} = e \rangle .$$

Definition (Coxeter complex[1])

For a Coxeter system (W, S) with $\#S = n$, the Coxeter complex is the hyperplane arrangement in R^n generated by the simple reflections, with angles between two hyperplanes given by $\pi / m(i,j)$, with $m(i,j)$ the order of $s_i s_j$.

The areas enclosed by hyperplanes are called the *chambers* of the complex.

The Coxeter group associated to $\mathrm{GL}_3(\mathbb{Q}_2)$ is $W := \mathbb{Z}^2 \rtimes S_3$. S_3 is the group of linear isometries of the triangle (figure 1). W then is the group of linear isometries plus discrete translations, giving a tessellation of the plane by equilateral triangles (figure 2). It is generated by three simple reflections.[1]

Visualizing buildings

Definition (Building[1])

A building is the union of Coxeter complexes called *apartments*. For any pair of chambers A, B there is an apartment containing both. For any two apartments that both contain A and B, there is an isomorphism between the two apartments.

Definition (Strong Transitivity[1])

A group is said to act strongly transitively on a building if it acts transitively on the chambers, and the stabilizer of any chamber acts transitively on the apartments.

If G acts strongly transitively on the building, the chambers correspond one-to-one to cosets of a subgroup $B \subset G$, which consists of matrices that are upper triangular modulo $T(2\mathcal{O}_{\mathbb{Q}_2})$. The main tool used in our visualization methods is the Bruhat decomposition $G = \coprod_{w \in W} BwB$.

Computational process

Generate chamber labels

Each chamber corresponds to a coset of B, which is given by a canonical representative through the Bruhat decomposition. We use this representative as the label for the chamber.

↓

Generate chamber graph

Two cosets gB and hB are adjacent if $Bg^{-1}hB = BsB$ for a simple reflection s.

↓

Visualization

The double coset BwB that the coset corresponding to a chamber lies in determines the horizontal position of the chamber. Chambers are then separated vertically based on the order of generation.

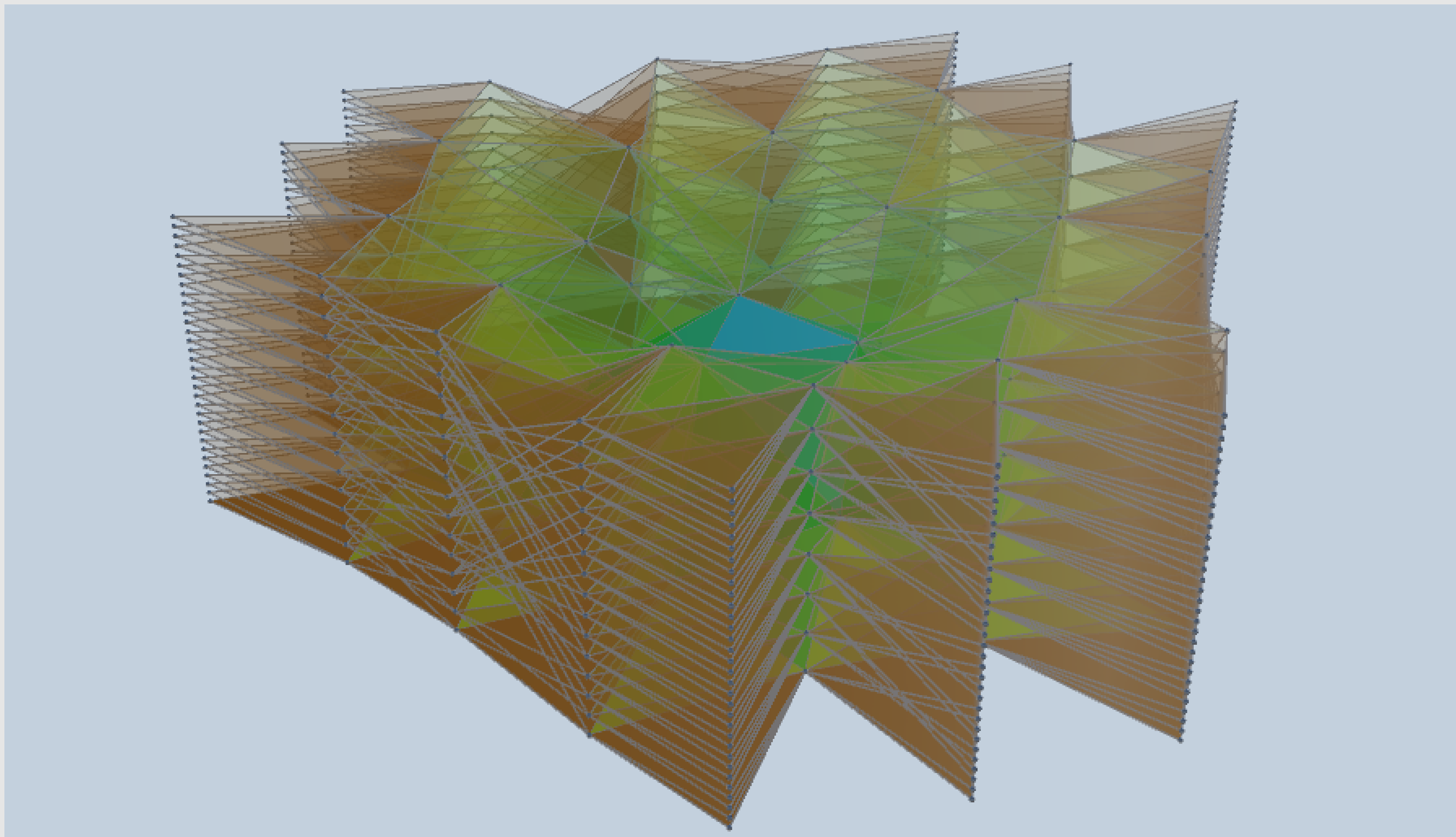

Figure 3: The Bruhat-Tits building of $\mathrm{GL}_3(\mathbb{Q}_2)$ up to distance 5 from the fundamental chamber

Conclusions/Future Work

The project has produced a solid proof of concept, to be found in the form of a web application at https://buildings.gallery. Based on the results, it is safe to say that there is a world of possibilities in visualizing buildings, and certainly it will lead to beautiful imagery useful for both outreach and education.

To increase the usefulness, methods for visualizing the different apartments and animating retractions can be developed. Another useful addition would be to add the functionality to draw galleries in the building, and animate where retractions take them.

The most obvious routes to take this research include optimizing the code for time efficiency and exploring the implications of changing the type of the building. For example, even affine type B_2 might present new difficulties.

Aside of this, a few basic methods are left unexplored. For example, the building discussed here can also be realized as the incidence geometry associated to the $\mathcal{O}_{\mathbb{Q}_2}$-lattices in $(\mathbb{Q}_2)^3$. This raises the question whether computing the chamber graph from this data would be more efficient.

Acknowledgments

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References

- P. Abramenko and K. S. Brown, *Buildings, Theory and Applications*. Springer, New-York, 2008.

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