Abstract
The aim of the project is to visualize the so called Bruhat-Tits Building of the group $GL_3(Q_2)$, with the intent of generalizing the methods for use with other building types.

Drawing the building is a three step process. Using both the Bruhat decomposition and the root group decomposition of this group, we obtain methods for generating all components (the chambers), deciding which are adjacent (generating the chamber graph) and finally positioning them in space.

Coxeter complexes

Definition (Coxeter system[1])
A Coxeter system is a group $W$ together with a set of generators $S$, called the set of simple reflections, such that $W$ allows a presentation:

$$W = \langle S \mid \forall i, j : s_i^2 = (s_i s_j)^{m(i,j)} = e \rangle.$$  

Definition (Coxeter complex[1])
For a Coxeter system $(W, S)$ with $\#S = n$, the Coxeter complex is the hyperplane arrangement in $\mathbb{R}^n$ generated by the simple reflections, with angles between two hyperplanes given by

$$\pi / \angle S(i,j),$$

with $m(i,j)$ the order of $s_i s_j$. The areas enclosed by hyperplanes are called the chambers of the complex.

The Coxeter group associated to $GL_3(Q_2)$ is $W := \mathbb{Z}^2 \times S_3$. $S_3$ is the group of linear isometries of the equilateral triangle (figure 1). $W$ then is the group of linear isometries plus discrete translations, giving a tessellation of the plane by equilateral triangles (figure 2). It is generated by three simple reflections [1].

Visualizing buildings

For a Coxeter system $(W, S)$, the Coxeter complex $\mathcal{C} = (\mathcal{P}, \mathcal{E})$ is a poset defined on the set of chambers $\mathcal{P}$ with a partial order relation $\mathcal{E}$ between two chambers $B_1$ and $B_2$ if and only if there is a sequence of chambers $B_{i-1} \rightarrow B_i$ with $i \in \mathbb{N}_0$. This allows a visualization $\mathcal{C}$ as a directed graph, so called the building $\mathcal{B} = (\mathcal{G}, \mathcal{E})$. Given a chamber $B$, the opposite of $B$ is its unique coset in $W$ such that $B \subset \mathcal{C}$ and $B \cap B = \mathcal{C}$. This allows a visualization of the building $\mathcal{B}$ as a directed graph, so called the Bruhat-Tits building $\mathcal{B} = (\mathcal{G}, \mathcal{E})$. The main tool used in our visualization methods is the Bruhat decomposition $G = \coprod_{w \in W} B \cdot w B$.

Computational process

Generate chamber labels
Each chamber corresponds to a coset of $B$, which is given by a canonical representative through the Bruhat decomposition. We use this representative as the label for the chamber.

Generate chamber graph
Two cosets $gB$ and $hB$ are adjacent if $\langle \langle g \rangle \rangle \cdot \langle \langle h \rangle \rangle^{-1} \cdot \langle \langle g \rangle \rangle = B$ for a simple reflection $s$.

Visualization
The double coset $B \cdot w \cdot B$ that the coset corresponding to a chamber lies in determines the horizontal position of the chamber. Chambers are then separated vertically based on the order of generation.

Conclusions/Future Work
The project has produced a solid proof of concept, to be found in the form of a web application at https://buildings.gallery. Based on the results, it is safe to say that there is a world of possibilities in visualizing buildings, and certainly it will lead to beautiful imagery useful for both outreach and education.

To increase the usefulness, methods for visualizing the different apartments and animating retractions can be developed. Another useful addition would be to add the functionality to draw galleries in the building, and animate where retractions take them.

The most obvious routes to take this research include optimizing the code for time efficiency and exploring the implications of changing the type of the building. For example, even affine type $B_i$ might present new difficulties.

Aside of this, a few basic methods are left unexplored. For example, the building discussed here can also be realized as the incidence geometry associated to the $O_{2n}$-lattices in $(\mathbb{Q}_2)^n$. This raises the question whether computing the chamber graph from this data would be more efficient.

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References