Hilbert Schemes and Monomial Ideals

Joseph Donato, Faustas Udrenas, Zijian Zhang, Zhan Jiang, Monica Lewis, Timothy Ryan

Laboratory of Geometry at Michigan

LOG(M)

Introduction

Goal

Given a polynomial $p(d)$ and positive integer N, find the number of monomial ideals I of the ring $\mathbb{C}[x_0, ...x_N]$, up to saturation, in $N+1$ variables such that the Hilbert polynomial of I is $p(d)$.

Notice that our problem boils down to two sub problems:
1. Given a polynomial $p(d)$, determine whether it can be Hilbert polynomial or not. (If so, we need to further determine whether this polynomial is "nice" or not.)
2. Given a "nice" Hilbert polynomial $p(d)$, find the number of saturated monomial ideals I with the Hilbert polynomial $p(d)$.

Motivation

1.To explore and development a direction to study Hilbert Schemes.
2. Develop some tools(e.g., approaches and coding) to solve the problem, which can be further utilized by mathematicians and physicists to do some related research.

Definition. *(Hilbert Function $h_I(d)$) Given a ring $R = \mathbb{C}[x_0, ...x_N]$ and a monomial ideal I of that ring, define the Hilbert funtion $h_I : \mathbb{Z} \longrightarrow \mathbb{Z}$ to be mapping an integer d to the number of degree d monomials which lie outside of the ideal I.*

Definition. *(Saturation)Consider ideal I, J in a ring R, I and J have the saturation if $I_d = J_d, \forall d >> 0$, where d is the degree of the monomials.*
Note: In this definition, $I_d = S_d \cap I$, where S_d denotes the set of monomials of degree d in this ring.

Background

Theorem 1: (Hilbert)

There is a polynomial H_I in the variable d such that $h_I(d)$ for $d >> 0$. H_I is called the Hilbert polynomial of I.
Note: $d >> 0$ means $\exists N$ such that $\forall d, d \geqslant N$ implies $h_I(d) = H_I(d)$.

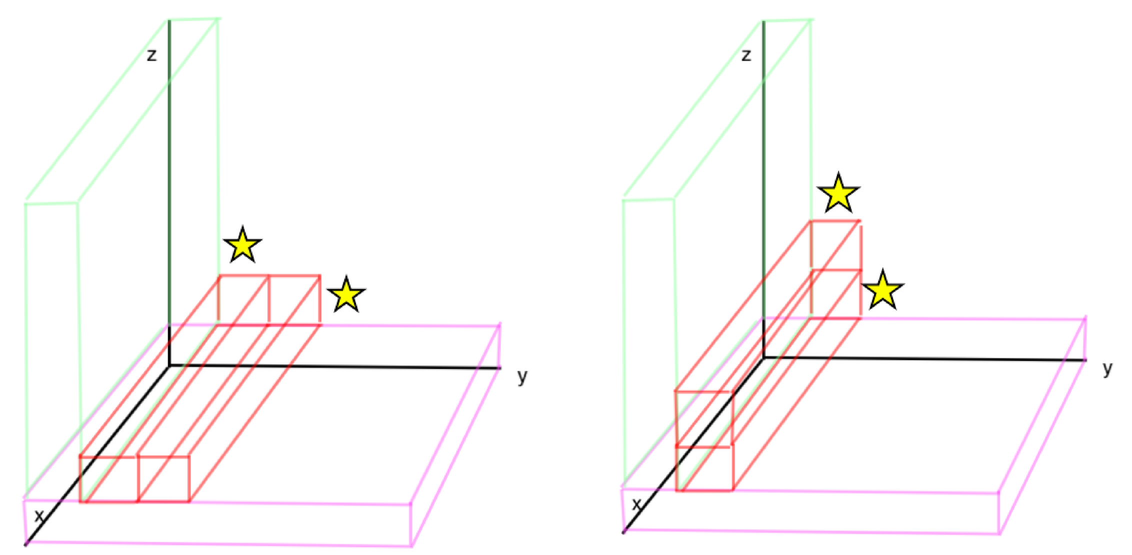
Theorem 2: (Macaulay)

Given $R = \mathbb{C}[x_0, ..., x_N]$ and polynomial $p(d)$ in one variable, there exists ideals in R with Hilbert polynomial p if and only if $p(d)$ can be written in the form $\Sigma_{i=1}^{m}\binom{d+\lambda_i-i}{\lambda_i-1}$ for some $N \geq \lambda_1 \geq ... \geq \lambda_m \geq 1$.

"Black Box": To count the number of saturated ideals, we can proceed from the λ-sequence found from Theorem 2. Suppose the λ-sequence is $\lambda = [(N)^{p_N}, (N-1)^{p_{N-1}}, ..., 1^{p_1}]$, where p_i denotes the number of i's in the λ-sequence, then the number of saturated ideals is the number of ways to lay down p_N N-planes, ..., p_i i-planes, ..., p_1 1-planes (i.e. rows) in $(N+1)$-space.

Black Box Example

Here we give a visual representation of what we mean by laying down P_i i-planes for some lambda sequence. Consider the polynomial $p(d) = 2d + 3$. This is in fact a Hilbert Polynomial (See below). The corresponding lambda sequence for $p(d)$ is $\lambda = (2, 2, 1, 1)$. Thus, we count the number of ways to place two 2-planes and two 1-planes (rows) in the xyz-grid.
Below are two examples of arranging these 2-planes and 1-planes in the xyz-grid.



We use the stars here to signify the monomials in this xyz-grid that make up the saturated monomial ideals we are counting.

Results

Here are the results we have been able to prove over the course of this project.

Theorem 3: (N=2 count)

For a given lambda sequence $\lambda = (2^{[p]}1^{[r]})$. The number of saturated ideals associated to the Hilbert polynomial whose lambda sequence is λ is

$$\binom{p+2}{2} \sum_{k_1+k_2+k_3=r}^{\binom{r+2}{2}} [f_2(k_1)f_2(k_2)f_2(k_3)]$$

Where $k_1, k_2, k_3 \geq 0$ and where f_2 is a function $f_2 : \mathbb{N} \longrightarrow \mathbb{N}$ which maps each natural number n to the number of integer partitions of n.

Clearly, in order to find the count, we first need to know the λ-sequence associated to a polynomial $p(d)$. This is generally difficult to do. However, for constant and linear polynomials, we know exactly when λ exists and what form it takes.

Theorem 4: (Lambda Criterion for Linear and Constant Polynomials)

Suppose $p(d)$ is a polynomial in d

- If $p(d) = Md - r$ for some $M, r \in \mathbb{R}$, $p(d)$ is a Hilbert polynomial if and only if $M \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}$ and

$$r \leq \frac{M^2 - 3M}{2}$$

and therefore λ takes the form

$$\lambda = (2^{[M]}, 1^{[\frac{M^2-3M}{2} - r]})$$

However, for higher order polynomials, finding λ is not as straightforward and we needed to approach the problem a different way. Thus we wrote code to tackle this issue. Now, we can always find a λ for any polynomial (if it exists). The code is further explained in the Methods section.

Methods

As one can see from theorem 4, determining whether or not a polynomial $p(d)$ is indeed a Hilbert polynomial and recovering a lambda sequence from it by hand becomes very tedious when we increase the degree of $p(d)$. As solution to this problem was designing an algorithm which takes in a polynomial $p(d)$ and outputs the lambda sequence associated with it if one exists.
The algorithm we designed is a brute force type algorithm which uses the polynomial interpolation theorem to identify whether or not $p(d)$ is a Hilbert polynomial and if its then it recovers the lambda sequence. Recall the polynomial interpolation theorem which states the following.

Theorem 5: (Polynomial Interpolation Theorem)

Given $n+1$ distinct real numbers $x_0, ..., x_n$ and $n+1$ arbitrary values $y_0, ..., y_n$, there is a unique polynomial p_n of degree n s.t.

$$p_n(x_i) = y_i \quad \forall i \in \{0, ..., n\}$$

Keeping this in mind, our algorithm goes through every possible lambda sequence up to a certain size which the user decides and with each lambda sequence it uses the expression in theorem 2 to see if it matches the values $y_0, ..., y_n$ which we generate before hand. The algorithm generates all possible lambda sequences of size m by conducting a weak composition of m into $n+1$ boxes where n is the degree of $p(d)$.

Future Directions

As of right now we are confident in counting the number of saturated ideals for when $N = 1$ and $N = 2$. However, when $N = 3$ counting the number of saturated ideals becomes very difficult since we are counting the number of ways to configure certain objects in $4 - D$ space. Given this, we currently plan on only counting the number of saturated ideals in $N = 3$ for the cases spelled out in the following theorem

Theorem 6: (Skjelnes and Smith)

Let $p(d)$ be a polynomial in a single variable with some lambda sequence $\lambda = (\lambda_1, ..., \lambda_r)$ with $n \geq \lambda_1 \geq ... \geq \lambda_r \geq 1$. The Hilbert Scheme is "nice" if at least one of the following is satisfied

- $n \leq 2$
- $\lambda_r \geq 2$
- $r \leq 1$ or $\lambda = (n^{r-2}, \lambda_{r-1}^1, 1^1)$ for all $r \geq 2$.
- $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 2^0, 1^1)$ for all $r \geq s$ and all $0 \leq s \leq r-3$.
- $\lambda = (n^{r-s-5}, 2^{s+4}, 1^1)$ for all $0 \leq s \leq r-5$ and all $r \geq 5$.
- $\lambda = (n^{r-3}, 1^3)$ for all $r \geq 3$.

References

[1] D. Hilbert, "Uber die Theorie der algebraischen Formen", Math. Annalen, 36 (1890), HSff.

[2] F.S MacAulay. Some Properties of Enumeration in the Theory of Modular Systems Proc. Lond. Math. Soc. (2), 26, pp. 531-555 (1927)

[3] R. Skjelnes and G. Smith - "Smooth Hilbert Schemes" preprint 2019