Growth Rates of Tent Maps

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Introduction

Goal

Deepen our understanding of the Thurston Set.

History of the Thurston Set

The Thurston Set, approximated in Figure 1, is a complex and beautiful image generated from a relatively simple dynamical system. Its namesake, the prolific William Thurston, created this set and was investigating it shortly before his unfortunate death in 2012 [5]. We aim to further research his conjectures and consider possible similarities between the Thurston Set and the better-understood Mandelbrot Set.

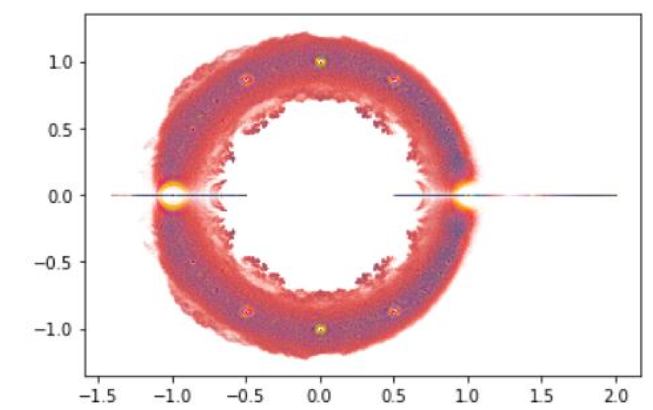

Figure 1: *Approximation of the Thurston Set plotted in $\mathbb{C}$ for superattracting β with orbit length ≤ 22, 4,190,079 points.*

Tent Maps

For $\beta \in (1,2]$, the β-*tent map* is the map $f : [0,1] \to [0,1]$ defined by

$$f_\beta(x) = \begin{cases} \beta x & \text{for } x \in [0, \frac{1}{\beta}) \\ 2 - \beta x & \text{for } x \in [\frac{1}{\beta}, 1]. \end{cases}$$

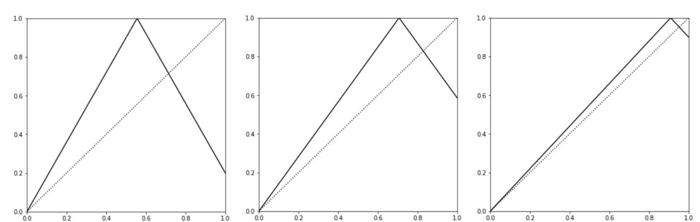

Figure 2: *Three examples of tent maps for $\beta = 1.8, 1.4, 1.1$, respectively.*

Superattracting

A function f_β is *superattracting* provided that there exists an $n \in \mathbb{N}$ such that $f_\beta^n(1) = 1$. We call the smallest n the *orbit length of* f_β. We can encode this value in an infinite sequence of 1's and 0's called an *itinerary*.

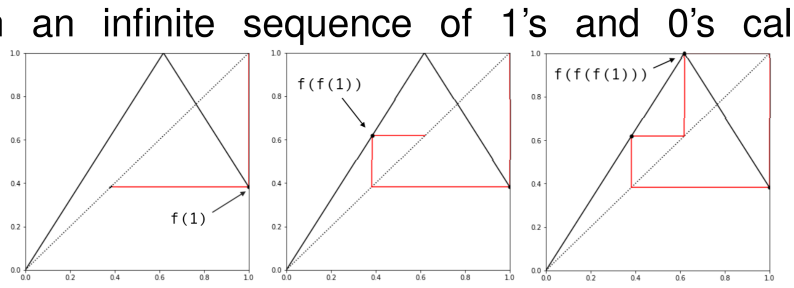

Figure 3: *Series of images showing iteration of f_φ, where φ is the Golden Ratio. We see f_φ is superattracting with orbit length 3 because $f_\varphi^3(1) = 1$.*

The Parry Polynomial and its β-Conjugates

Given a superattracting value of β, we can use the itinerary to create a unique polynomial, P_β, called the *Parry Polynomial*. *β-conjugates* of a given β are all the roots of P_β, including β itself. For a superattracting β, all its β-conjugates are elements of the Thurston Set [1].

Thurston Set

The *Thurston Set* is the closure of the set of β-conjugates for all superattracting beta.

Our Findings

Distribution of Superattracting β within $(1,2]$

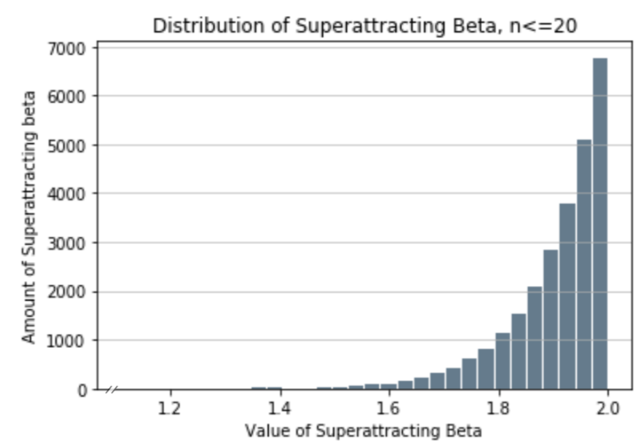

Figure 4: *A histogram showing the distribution of superattracting β within the interval $(1,2]$.*

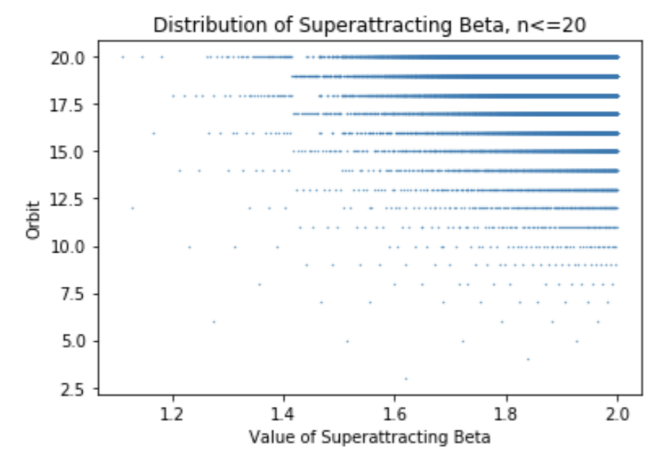

Figure 5: *This image shows the connection between orbit length and the value of β. The effects of period-doubling [1] are visible here.*

The Thurston Set and Master Teapot

Figure 6 on the right shows our approximation of Thurston's Master Teapot. We created this image by plotting the β-conjugates of all superattracting β with orbit length 20 or less against their associated β values. The Thurston Set (Figure 1) is the projection of the Master Teapot onto the complex plane. Our program plotted 1,046,463 total points.

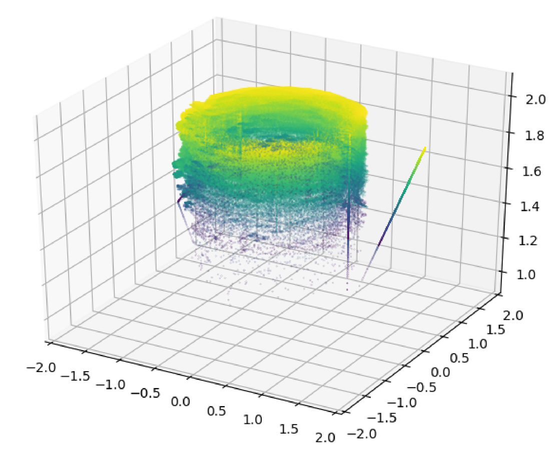

Figure 6: *Approximation of Thurston's Master Teapot plotted in $\mathbb{C} \times \mathbb{R}$.*

Analogous Mandelbrot-Julia Correspondence

Inspired by a question posed by Lindsey and Wu, we investigate the analogous correspondence between the Thurston Set and its attracting sets. By using the iterated function system constructed in other literature [2][3], we create the attracting sets for various points along the boundary of the Thurston Set. As can be seen in Figures 7 and 8 below, there is evidence to support a correspondence like that of the Mandelbrot Set.

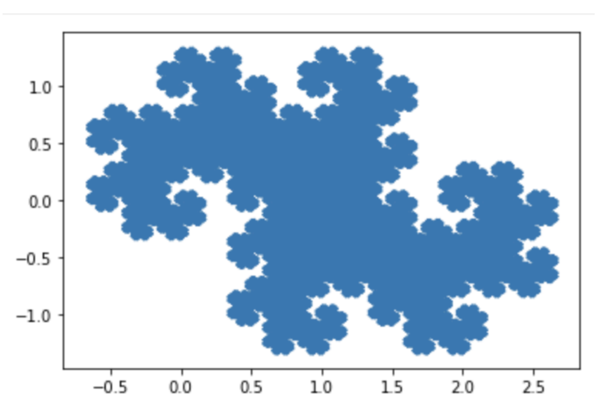

Figure 7: *Attracting set for $c = .5 + .5i$.*

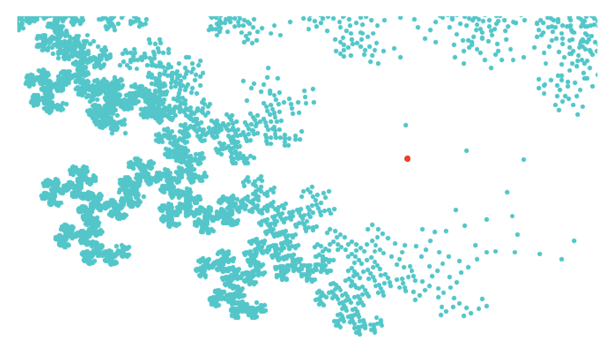

Figure 8: *Thurston Set zoomed in around $c = .5 + .5i$.*

Computability

Admissibility Criterion [4]

A finite string $w = (x_1, ..., x_n)$ of zeros and ones is admissible (an itinerary of some superattracting β) if and only if:

- $\sum_{i=1}^n x_i$ is even (i.e. w has "positive cumulative sign").
- $x_{k+1}, ..., x_n, x_1, ..., x_k \leq_{twist} (x_1, ..., x_n)$ for all $k = 1, ..., n$.

Code

- *Encode the itinerary*
  Let $w_n = (x_0, x_1, ..., x_n)$ be an itinerary sequence, let $x_{n_0}, x_{n_1}, ..., x_{n_{k-1}}$ be all the digits in w_n such that $x_{n_i} = 1$. We encode w_n as $S_{w_n} = (n_0, n_1, ..., n_{k-1})$, where S_{w_n} keeps track of all the index of ones in w_n.For example, let $w_n = (1,0,1,1)$. Since $x_0 = x_2 = x_3 = 1$, w_n is encoded as $S_{X_n} = (0,2,3)$.
- *Multi-threading*
  In order to make our code more efficient, we worked with Samuel Hansen to learn multi-threading. We increased our data from superattracting β with orbit length 20 to orbit length 33 using the computers in the Digital Project Studio.
- *CSV*
  We created a CSV file that stores the itineraries of all superattracting β and their orbit lengths up to orbit length 33, around 12 GB of data. Storing this data allows us to speed up our code because we simply have to read the values now instead of recalculating every time we need to access this data.

Iterated Function System (IFS)

An IFS is a method of constructing fractals that randomly composes simpler contraction functions to build a self similar fractal. Our iterated functions system was a set of two functions,

$$\begin{cases} f_0(x) = cx \\ f_1(x) = 2 - cx \end{cases}$$

for fixed $c \in \mathbb{C}$. To create Figure 7, we randomly composed these two functions 100 times in 10,000 different orders, evaluated at 0, and plotted the output of each. Depending on the order of composition, the contractions will interact in slightly different ways but ultimately be attracted to this shape, hence the name *attracting set*.

Future Directions

Make CSV Public

- Further optimize code. Run code on supercomputers to generate a list of elements in the Thurston Set.
- Publish a sorted list of elements in the Thurston Set and their corresponding itineraries.

More on Analogous Mandelbrot-Julia Correspondence

- Investigate if there is such a correspondence for "slices" of the Master Teapot. Instead of composing the two functions of the IFS randomly, we would do so in a specific order according to the itinerary of a fixed superattracting β and all itineraries that are less than it per the $\leq_{twist}$ lexicographical ordering.
- Understand and explore Lindsey-Wu's new characterization of the inside of the Thurston set.

References

[1] Harrison Bray, Diana Davis, Kathryn Lindsey, Chenxi Wu. The Shape of Thurston's Master Teapot. (Preprint) 2019.

[2] Danny Calegari, Sarah Koch, and Alden Walker. Roots, Schottky Semigroups, and a Proof of Bandt's Conjecture. *Ergodic Theory Dynam. Systems.* 2017.

[3] Kathryn Lindsey, Chenxi Wu. A characterization of Thurston's Master Teapot. (Preprint) 2019.

[4] John Milnor and William P. Thurston. On iterated maps of the interval. Dynamical Systems, 1342:465â 563, 1988.

[5] William Thurston. Entropy in Dimension One. *Frontiers in Complex Dynamics.* 2014.

[6] Giulio Tiozzo. Galois Conjugates of Entropies of Real Unimodal Maps. *International Mathematics Research.*. 2018.