ON AN ANALYTIC SOLUTION TO A 2-D PDE TYPE OF AN IDEAL FLUID

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SYNOPSIS

This PDE system, whose cardinality of equations exceed that of its unknowns, is a 2-D equivalent of a general 3-D system, investigating a particular kind of a 2-D PDE of an ideal fluid. It delineates the thermal motion of poly-tropic gas with constant density. Isothermal gas motion with an adiabatic index different to the unit is reduced to the same system. Its study is less complicated in special Lagrangian coordinates. The resulting system consists of linear equations, solved analytically.

SYSTEM PASSIVITY

The problem is considered locally, all functions are differentiable any number of times. By the system's compatibility, it has a nonempty set of solutions. Introducing the complex-valued function $z = x + iy$, system (2) becomes: $z_\xi = iz_{tt}, \frac{i}{2}(z_\xi \overline{z}_\eta - \overline{z}_\xi z_\eta) = 1$ This separation is so as to make the systems passive. A torus system is considered instead as its consequence. A 2-D vector space bi-linear skew-symmetric form is introduced. For $\mathbf{a} = (a^1, a^2)$ and $\mathbf{b} = (b^1, b^2)$, set

$$\mathbf{a} \vee \mathbf{b} = \begin{vmatrix} a^1 & b^1 \\ a^2 & b^2 \end{vmatrix}.$$

THE SYSTEM

$u_t + uu_x + vu_y + p_x = 0,$
$v_t + uv_x + vv_y + p_y = 0,$
$u_x + v_y = 0, p_t + up_x + vp_y = 0.$ (1)
Where t is the time, x, y are spatial coordinates, u, v are the velocity components, p is the pressure deviation from the set value p_0. The coordinates are reduced for constant pressure. The derived system consists of equations: $x_\xi = -y_{tt}$, $y_\xi = x_{tt} x_\xi y_\eta - x_\eta y_\xi = 1.$ (2) Note that this transformation to Lagrangian coordinates is viable, if p is constant only.

RESULTING SYSTEM

It follows that the vector $\mathbf{z} = (z^1, z^2) = (z_\xi, z_\eta)$ fulfills: $\mathbf{z}_\xi = i\mathbf{z}_{tt}, \frac{i}{2}\, \mathbf{z} \vee \overline{\mathbf{z}} = 1.$
$\alpha = \frac{i}{2}\, \mathbf{z} \vee \overline{\mathbf{z}}, \beta = \frac{1}{2}\left(\mathbf{z}_t \vee \overline{\mathbf{z}} - \mathbf{z} \vee \overline{\mathbf{z}}_t\right),$
$\gamma = -\frac{i}{2}\left(\mathbf{z}_{tt} \vee \overline{\mathbf{z}} - 2\, \mathbf{z}_t \vee \overline{\mathbf{z}}_t + \mathbf{z} \vee \overline{\mathbf{z}}_{tt}\right),$
$\delta = -\frac{1}{2}\left(\mathbf{z}_{ttt} \vee \overline{\mathbf{z}} - 3\, \mathbf{z}_{tt} \vee \overline{\mathbf{z}}_t\right.$
$\left.+3\, \mathbf{z}_t \vee \overline{\mathbf{z}}_{tt} - \mathbf{z} \vee \overline{\mathbf{z}}_{ttt}\right),$
$\varepsilon = \frac{i}{2}\left(\mathbf{z}_{tttt} \vee \overline{\mathbf{z}} - 4\, \mathbf{z}_{ttt} \vee \overline{\mathbf{z}}_t\right.$
$\left.+6\, \mathbf{z}_{tt} \vee \overline{\mathbf{z}}_{tt} - 4\, \mathbf{z}_t \vee \overline{\mathbf{z}}_{ttt} + \mathbf{z} \vee \overline{\mathbf{z}}_{tttt}\right).$ (4) The functions $\alpha, \beta, \gamma, \delta, \varepsilon$ are selected such that they are real and the following relations are true: $\alpha_\xi + \beta_t = \beta_\xi + \gamma_t = \gamma_\xi + \delta_t = \delta_\xi + \varepsilon_t = 0$. It should be noted that the systems obtained can be transformed to equivalent passive orthonormal systems, but that would complicate the formulas.

CONTINUOUS TRANSFORMATIONS GROUP BY OPERATORS

It is known, that the solution to system (2) cannot depend on an arbitrary function of two variables and has at most four arbitrary single-variable functions. A solution of an analog of system (2) is known too. System (1) is relevant to research on thermonuclear fusion. Here, two classes of solutions are investigated and it is stated, without proof, that other solutions do not exist. System (2) admits a group of continuous transformations generated by operators:

$X_1 = \varphi(\eta)\partial_\xi$, $X_2 = \partial_\eta$, $X_3 = t\partial_x$, $X_4 = t\partial_y$, $X_5 = \partial_x$, $X_6 = \partial_y$, $X_7 = \partial_t$, $X_8 = -y\partial_x + x\partial_y$, $X_9 = t\partial_t + 2\xi\partial_\xi - 2\eta\partial_\eta$, $X_{10} = x\partial_x + y\partial_y + 2\eta\partial_\eta$. (3)

LEMMAS & SYSTEM COMPATIBILITY

Lemma 1 $\Delta_1 = -4(\alpha\alpha_{tt} - \alpha_t^2 + \alpha\gamma - \beta^2), \Delta_2 = 2(\alpha\alpha_{ttt} - \alpha_t\alpha_{tt} + \alpha\gamma_t + \gamma\alpha_t - 2\beta\beta_t),$ $\Delta_3 = 2(\beta\alpha_{tt} - 2\alpha_t\beta_t + \alpha\beta_{tt} + \alpha\delta - \beta\gamma), \Delta_4 = -\alpha_t\alpha_{ttt} + \alpha_{tt}^2 - \alpha_t\gamma_t + \beta\beta_{tt} + \beta\delta - \gamma^2,$ $\Delta_5 = -\beta\alpha_{ttt} + 2\beta_t\alpha_{tt} - \alpha_t\beta_{tt} - \alpha_t\delta + 2\gamma\beta_t - \beta\gamma_t$. *Then the identity is true.* $\Delta_1\mathbf{z}_{tt} + (\Delta_2 + i\Delta_3)\mathbf{z}_t + (\Delta_4 + i\Delta_5)\mathbf{z} = 0.$ *(5) Moreover,* $\Delta_1 \neq 0$ *is equivalent to the condition* $\mathbf{z}_t \vee \mathbf{z} \neq 0$. *By the system compatibility of previous equations,* $S + \overline{S} = S_t = S_\xi = 0.$

Lemma 2 $\Omega = (\alpha_{tttt} + 2\gamma_{tt} + \varepsilon)(\beta^2 - \alpha(\gamma + \alpha_{tt}) + \alpha_t^2) + (\alpha_{tt} + \gamma)(4\beta_t^2 + (\alpha_{tt} - \gamma)^2) + \alpha(\beta_{tt} + \delta)^2 + \alpha(\alpha_{ttt} + \gamma_t)^2 + 2(\beta_{tt} + \delta)(\alpha_{tt}\beta - 2\alpha_t\beta_t - \beta\gamma) + 2(\alpha_{ttt} + \gamma_t)(\alpha_t\gamma - \alpha_{tt}\alpha_t - 2\beta_t\beta) = 0.$ *(6)*

Lemma 3 $\beta_t = \gamma_{tt} = \delta_{ttt} = \varepsilon_{tttt} = 0.$ *(7)* $(\beta^2 - \gamma)\varepsilon + \delta^2 + \gamma_t^2 - 2\beta\gamma\delta + \gamma^3 = 0$ *(8) If equations (7),(8), then* $\gamma_t = 0$. $\gamma - \beta^2 \neq 0$. $z_{tttt} = 2iKz_{ttt} + Tz_{tt}, z_{tt\eta} = 2iKz_{t\eta} + Tz_\eta, -\frac{1}{2}(z_{tt}\overline{z}_\eta + \overline{z}_{tt}z_\eta) = 1, z_{ttt}z_\eta - z_{tt}z_{t\eta} \neq 0$. *Under the previous condition and system, it is compatible only if* $\delta_t = 0$.

Lemma 4 *Let* $z_{tt} \neq 0$, *the functions* K *and* T *do not depend on* t, *and the previous system. Then for* $K_\eta \neq 0$, *each of its solutions fulfills equations* $z_{ttt} = iNz_{tt}, z_{t\eta} = iNz_\eta - iN^{-2}N_\eta z_{tt}.$, *with* N *a real function of* η. *If function* K *is a constant, then* T *is constant too.*

Lemma 5 *Let* $z_{tt} \neq 0$, *then the previous system is compatible only for* $k = m = 0$.

THEOREM SKETCH

Combining the sets of solutions of all passive sub-systems yields the final system solution set. All solutions depend only on i, not on t, nor j.

EPILOGUE & ONGOING/FUTURE DIRECTIONS

For all solutions to (5),(6) the curve's shape $\eta = const$ is invariant in time. Thus, any of these curves can be a moving solid wall. 3-D systems are transformable. Yet, the PDE system is still complex, its solution search in progress, and so to be reported in due course.

MAIN REFERENCE

Zel'Dovich, Y. B. & Raizer, Y. P. (2002). Physics of shock waves and high-temperature hydrodynamic phenomena.