ON AN ANALYTIC SOLUTION TO A 2-D PDE TYPE OF AN IDEAL FLUID

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SYNOPSIS
This PDE system, whose cardinality of equations exceed that of its unknowns, is a 2-D equivalent of a general 3-D system, investigating a particular kind of a 2-D PDE of an ideal fluid. It delineates the thermal motion of poly-tropic gas with constant density. Isothermal gas motion with an adiabatic index different to the unit is reduced to the same system. Its study is less complicated in special Lagrangian coordinates. The resulting system consists of linear equations, solved analytically.

SYSTEM PASSIVITY
The problem is considered locally, all functions are differentiable any number of times. By the system’s compatibility, it has a nonempty set of solutions. Introducing the complex-valued function \( z = x + iy \), system (2) becomes: \( z = iz_{1}, \frac{1}{2}(z_{1}^{2} - \tau_{x}^{2}z_{y}) = 1 \). This separation is so as to make the systems passive. A torus system is considered instead as its consequence. A 2-D vector space bi-linear skew-symmetric form is introduced. For \( a = (a^{1}, a^{2}) \) and \( b = (b^{1}, b^{2}) \), set
\[
\langle a, b \rangle = \begin{bmatrix} a^{1} & b^{1} \\ a^{2} & b^{2} \end{bmatrix}.
\]

THEOREM SKETCH
Combining the sets of solutions of all passive sub-systems yields the final system solution set. All solutions depend only on \( i \), not on \( t \), nor \( j \).

SYSTEMS THEOREM
The complex-valued function reduced to the same system. Its study is the motion of poly-tropic gas with arbitrary single-variable functions. A solution of an analog of system (2) is known too. System (1) is relevant to research on thermoclinic fusion. Here, two classes of solutions are investigated and it is stated, without proof, that other solutions do not exist. System (2) admits a group of continuous transformations generated by operators:
\[
X_{1} = \varphi(\eta)\partial_{\eta}, \quad X_{2} = \partial_{x}, \quad X_{3} = x = -y\partial_{x} + x\partial_{y}, \quad X_{9} = t\partial_{x} + 2\xi\partial_{y} - 2\eta\partial_{x}, \quad X_{10} = x\partial_{x} + y\partial_{y} + 2y\partial_{x},
\]

RESULTS & SYSTEM COMPATIBILITY
Lemma 1 \( \Delta_{1} = -4(\alpha_{1}\alpha_{2} - \alpha_{1}\gamma_{1} + \alpha_{2}\gamma_{1} + \gamma_{1}\gamma_{2}) \), \( \Delta_{2} = 2(\alpha_{1}\alpha_{2} + \alpha_{1}\gamma_{1} + \alpha_{2}\gamma_{1} - 2\beta_{1}\gamma_{2}) \), \( \Delta_{3} = 2(\beta_{1}\alpha_{2} + \alpha_{1}\beta_{1} + \alpha_{2}\beta_{1} - 2\alpha_{1}\gamma_{2}) \), \( \Delta_{4} = -2(\alpha_{1}\beta_{1} + \alpha_{2}\beta_{1} + \beta_{1}\gamma_{2} - \gamma_{1}\gamma_{2}) \). Then the identity is true. \( \Delta_{1}z_{tt} + (\Delta_{2} + i\Delta_{3})z_{t} + (\Delta_{4} + i\Delta_{5})z = 0 \). Moreover, \( \Delta_{1} \neq 0 \) is equivalent to the condition \( z_{t} \neq z \). By the system compatibility of previous equations, \( S + \frac{\eta}{\epsilon} = S_{z} = 0 \).

LEMMA 2 \( \Omega = (\alpha_{1}\alpha_{2} + 2\gamma_{1}\gamma_{3} + \varepsilon)(\beta^{2} - \alpha(\alpha_{1} + \alpha_{2}) + \alpha^{2} + \alpha(\gamma_{1} + \gamma_{2} + \gamma_{3}) + \gamma_{1}(\alpha^{2} + \gamma_{2} + \gamma_{3}) + \gamma_{2}(\alpha^{2} + \gamma_{1} + \gamma_{3}) + \gamma_{3}(\alpha^{2} + \gamma_{1} + \gamma_{2})). \)

LEMMA 3 \( \beta_{1} = \gamma_{1} = \Delta_{1}z_{tt} = \partial_{t}\nu_{tt} = 0. \) If equations (7) are solved, then \( \gamma_{1} = 0 \). \( \gamma_{2} \neq 0 \). \( \nu_{tt} = 2Kz_{tt} + Tz_{tt}, \quad \nu_{t} = 2Kz_{tt} + Tz_{tt}, \quad \Delta_{1}z_{tt} = Tz_{tt} - Tz_{tt} = Tz_{tt} = 1, \quad \Delta_{1}z_{tt} = \nu_{tt} \neq 0 \). Under the previous condition and system, it is compatible only if \( \delta_{1} = 0 \).

LEMMA 4 Let \( z_{t} \neq 0 \), the functions \( K \) and \( T \) do not depend on \( t \), and the previous system. Then for \( K \neq 0 \), each of its solutions fulfills equations \( z_{tt} = iNz_{tt}, \quad z_{tt} = iNz_{tt}, \quad z_{tt} = iNz_{tt} - iN^{2}Nz_{tt}, \), with \( N \) a real function of \( t \). If function \( K \) is a constant, then \( T \) is constant too.

LEMMA 5 Let \( z_{tt} \neq 0 \), then the previous system is compatible only for \( k = m = 0 \).

EPilogue & ONGOING/FUTURE DIRECTIONS
For all solutions to (5),(6) the curve’s shape \( \eta = \text{const} \) is invariant in time. Thus, any of these curves can be a moving solid wall. 3-D systems are transformable. Yet, the PDE system is still complex, its solution search in progress, and so to be reported in due course.

MAIN REFERENCE