Shallow Water Models with Vertical Velocity Profiles

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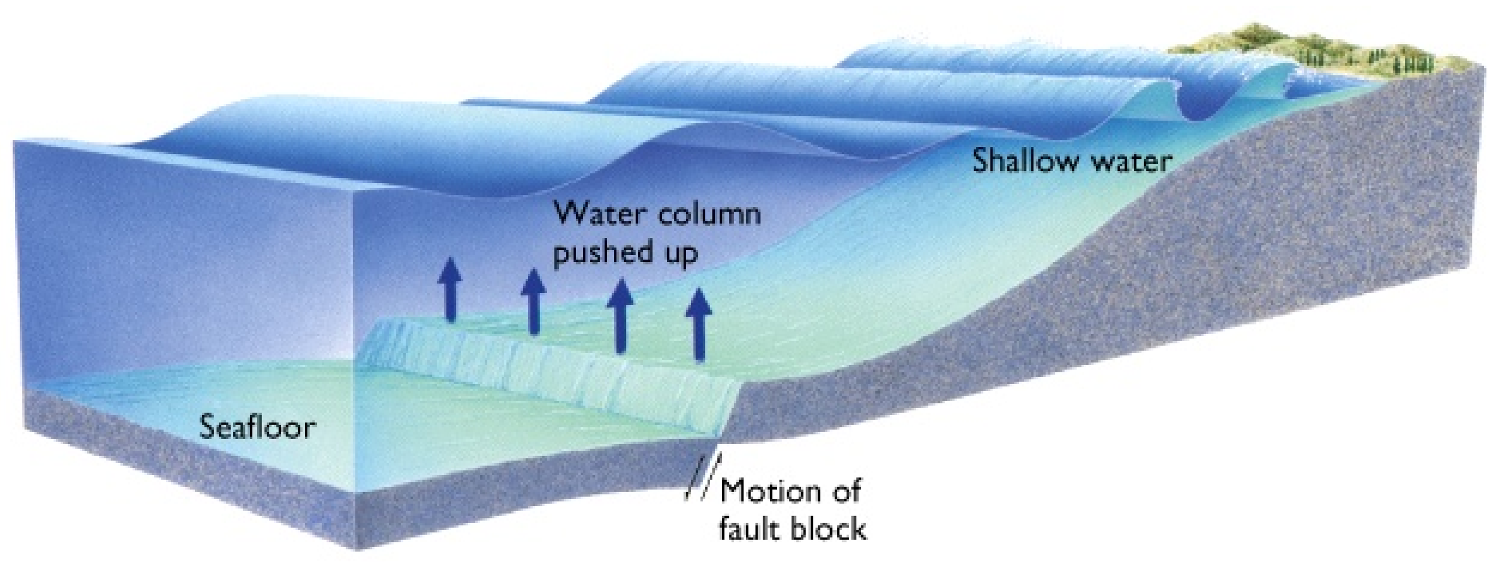
Application: Shallow Flows



Relevant scale is the **shallowness**

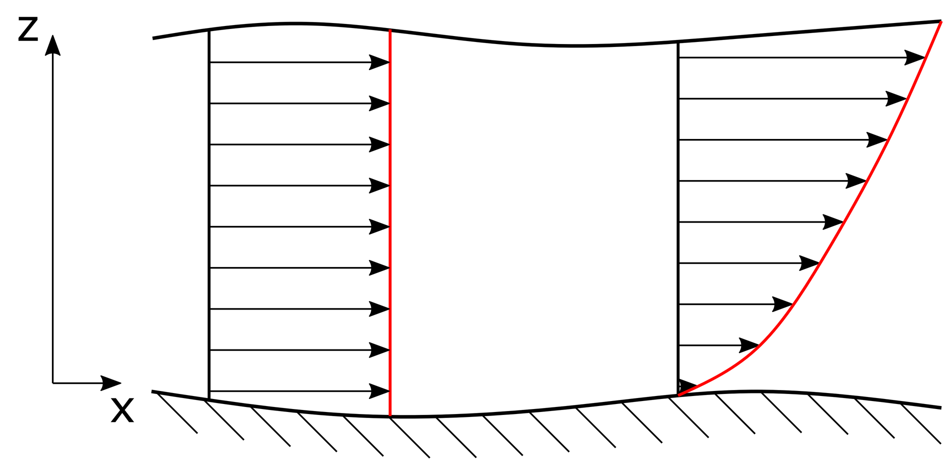
$$S = \frac{\text{water height}}{\text{wave length}} = \frac{h}{\lambda}$$

Model equation is the incompressible Navier-Stokes Equation

$$\nabla \cdot \boldsymbol{u} = 0, \quad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} + g$$

Ansatz: Polynomial velocity profile [1]

A constant velocity profile (as common for shallow water equations) is an unphysical simplification.



$$u(t,x,z) = \underbrace{u_m(t,x)}_{\text{mean of } u} \quad \text{vs} \quad u(t,x,z) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x)\phi_i\left(\frac{z-h_b}{h}\right)$$

Use **Legendre basis polynomials** ϕ_i, which are orthogonal on $[h_b(x), h(t,x)]$.
Projection onto test functions leads to moment model

$$\partial_t \boldsymbol{w} + \partial_x \boldsymbol{F}(\boldsymbol{w}) = \boldsymbol{Q}\partial_x \boldsymbol{w} + \boldsymbol{S}, \quad \boldsymbol{w} = (h, hu_m, h\alpha_1, \ldots, h\alpha_M)^T$$

Equations 1: Shallow Water Moment Model [1]

Example: $N = 2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = \boldsymbol{Q}\partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \frac{\nu}{\lambda}\boldsymbol{P}$$

with non-conservative part $\boldsymbol{Q}$ and friction term $\boldsymbol{P}$

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \text{ and } \boldsymbol{P} = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1) \\ 5(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2) \end{pmatrix}$$

Equations 2: Hyperbolic regularization [2]

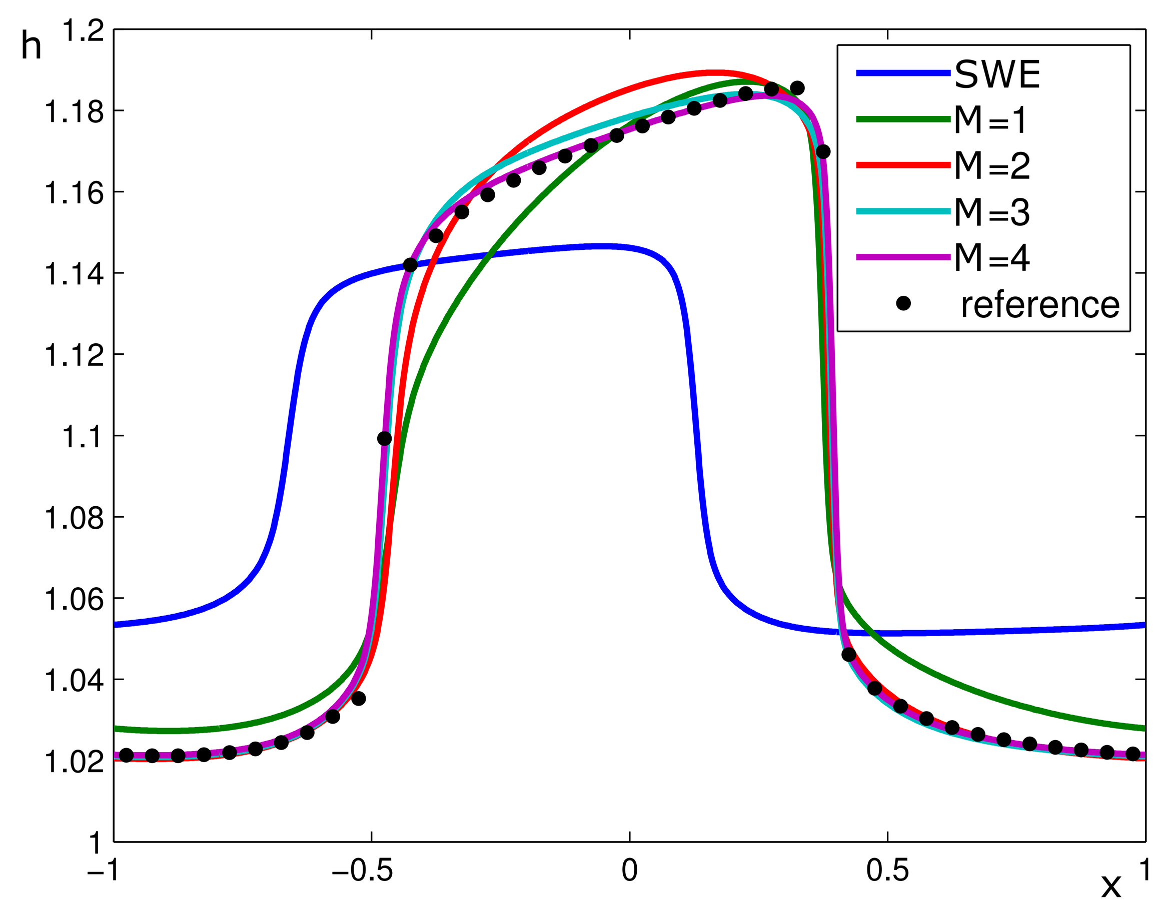
Model is **not hyperbolic** and requires regularization: $\partial_t \boldsymbol{w} + \boldsymbol{A}\partial_x \boldsymbol{w} = \boldsymbol{S}$

$$\boldsymbol{A} = \begin{pmatrix} & 1 & & & \\ gh - u_m^2 - \frac{1}{3}\alpha_1^2 & 2u_m & \frac{2}{3}\alpha_1 & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \ddots & \\ -\frac{2}{3}\alpha_1^2 & & \ddots & \ddots & \frac{M+1}{2M+1}\alpha_1 \\ & & & \frac{M-1}{2M-1}\alpha_1 & u_m \end{pmatrix}$$

Regularized model is **hyperbolic** with **real eigenvalues**

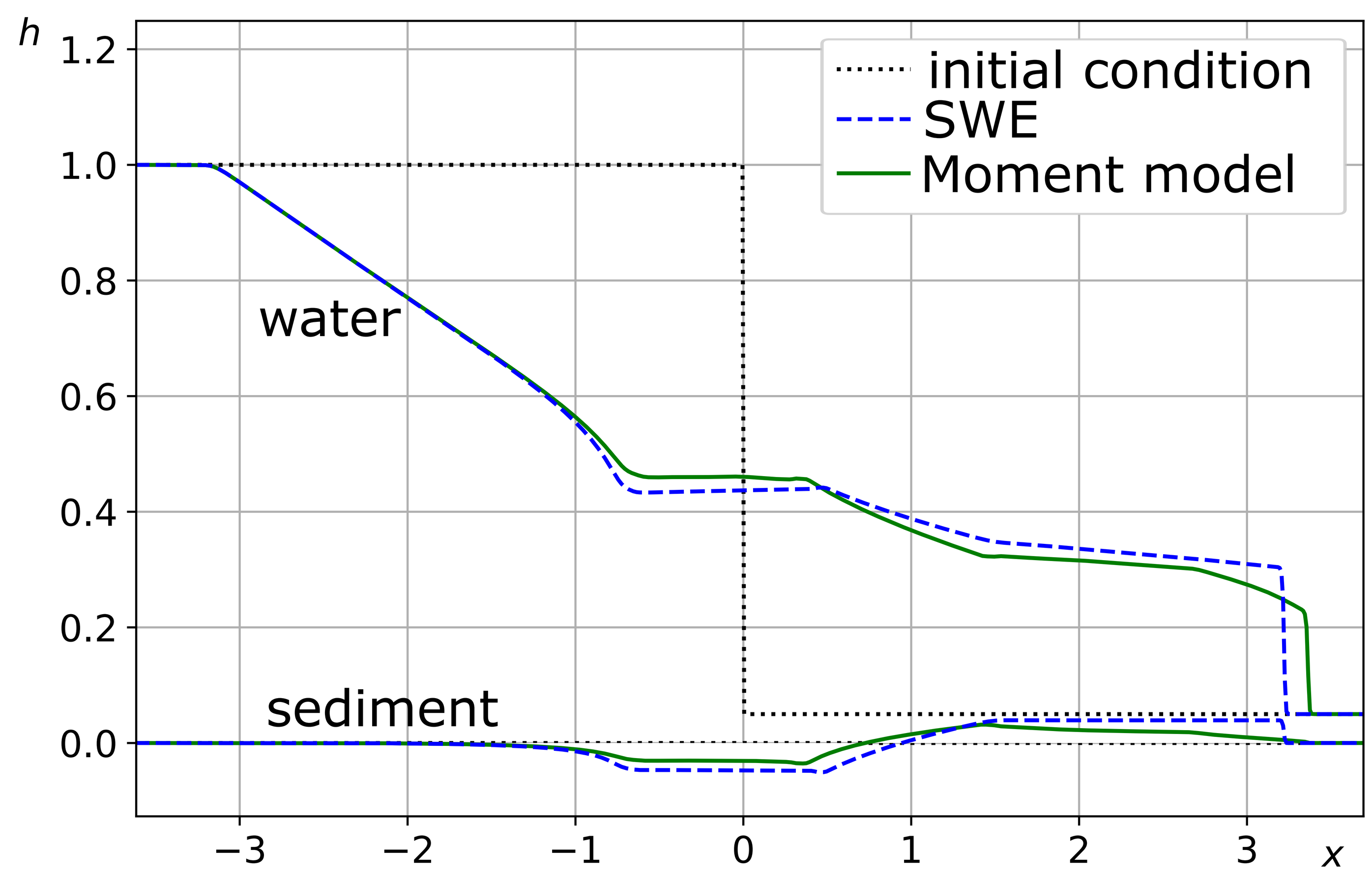
$$\lambda_{1,2} = u_m \pm \sqrt{gh + \alpha_1^2}$$
$$\lambda_{i+2} = u_m + b_i \cdot \alpha_1, \quad i = 1, \ldots, M$$

Results 1: Smooth wave transport[1, 2]



Hyperbolic moment model **converges to reference solution** with increasing M.

Results 2: Shallow sediment transport [5]



Vertical velocity profile of moment model yields **realistic bottom sediment transport**.

Further work:

(1) equilibrium stability analysis [3]
(2) well-balanced numerical schemes [4]
(3) Savage-Hutter friction term

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References

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[2] J. Koellermeier, M. Rominger. Analysis and Numerical Simulation of Hyperbolic Shallow Water Moment Equations, *Commun. Comp. Phys.*, (2020)

[3] Q. Huang, J. Koellermeier, W.-A. Yong. Equilibrium Stability Analysis of Hyperbolic Shallow Water Moment equations, *submitted*

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[5] J. Garres-Díaz, T. Morales de Luna, M. J. Castro, J. Koellermeier. Shallow Water Moment models for bedload transport problems, *to appear in Commun. Comp. Phys.*