Stability of a Nonlocal Conservation Law Modeling Traffic Flow

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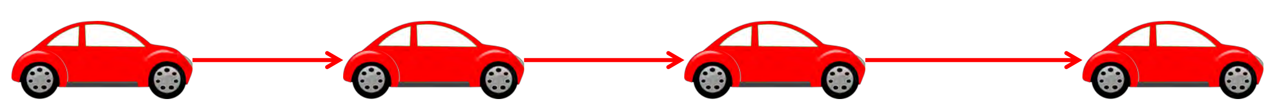
Main Message

- **Traffic flow of connected vehicles can be modeled by nonlocal conservation laws.**
- **Improper use of nonlocal information in the vehicle velocity selection could result in persistent traffic waves.**
- **To utilize the benefits of vehicle connectivity, nearby information should deserve more attention.**

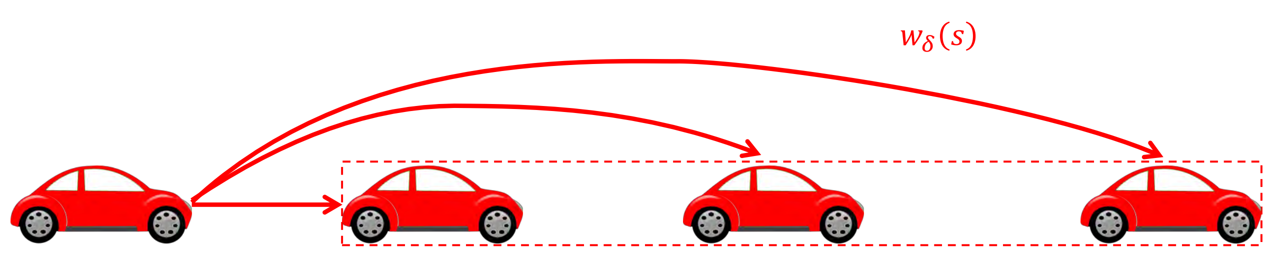
Modeling Traffic Flow: Non-Connected to Connected



LWR (1956)[1,2]:

$$\partial_t \rho(x,t) + \partial_x \left(\rho(x,t) U\left(\rho(x,t)\right)\right) = 0.$$

- $\rho(x,t)$: traffic density;
- $u(x,t) = U(\rho(x,t))$: traffic velocity;
- Capture shock waves (traffic jams).



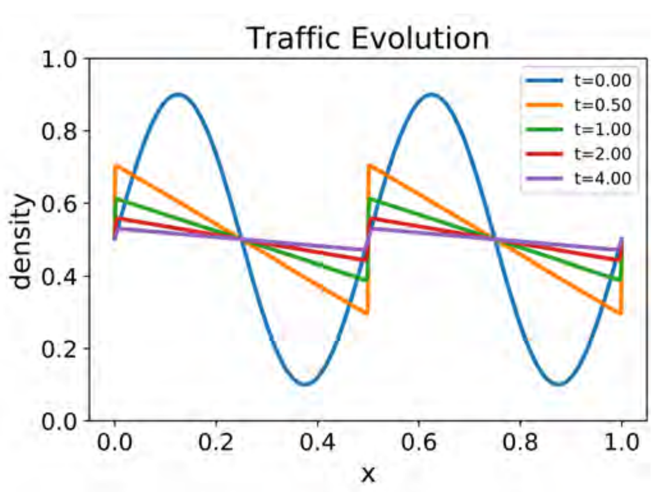
Nonlocal LWR (2016)[3]:

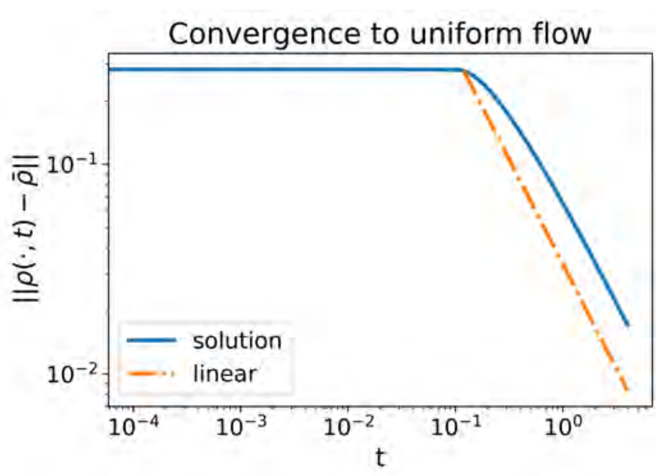
$$\partial_t \rho(x,t) + \partial_x \left(\rho(x,t) U\left(\int_0^\delta \rho(x+s,t) w_\delta(s)\, ds\right)\right) = 0.$$

- $\int_0^\delta \rho(x+s,t) w_\delta(s)\, ds$: nonlocal traffic density;
- $w_\delta(s)$ characterizes how nonlocal information is used;
- This is a nonlocal conservation law.

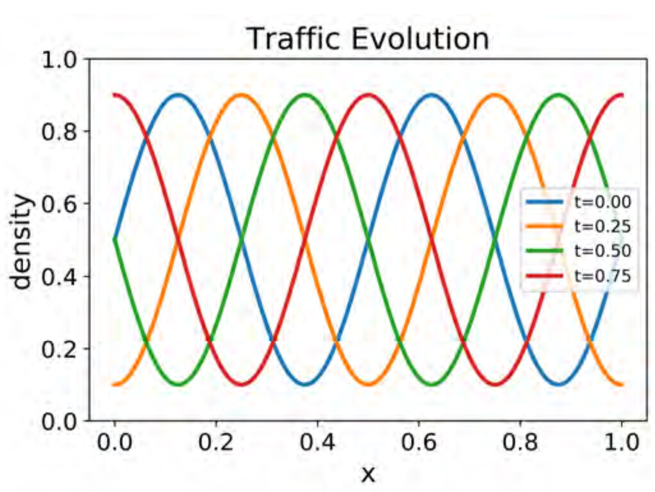
Connectivity: Benefit or Drawback

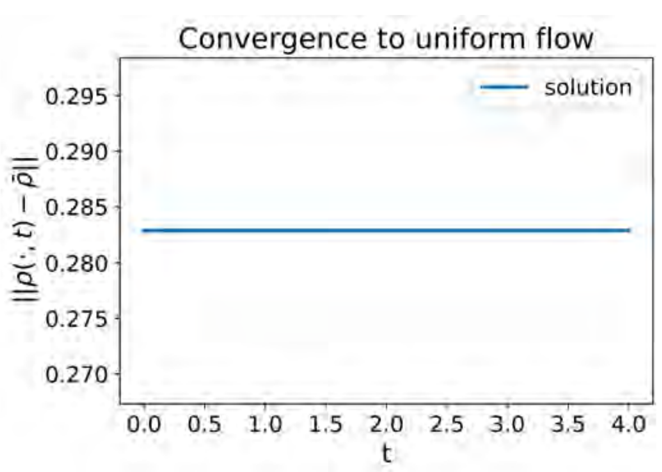
- Set $U(\rho) = 1 - \rho$, $\rho_0(x) = 0.5 + 0.4\sin(4\pi x)$.
- Local LWR:



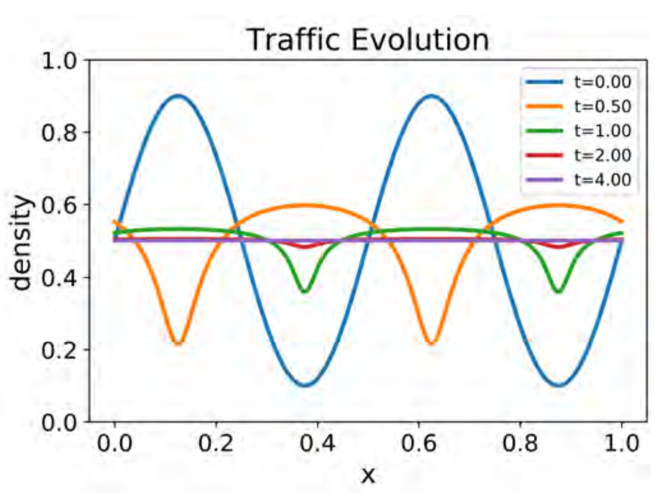


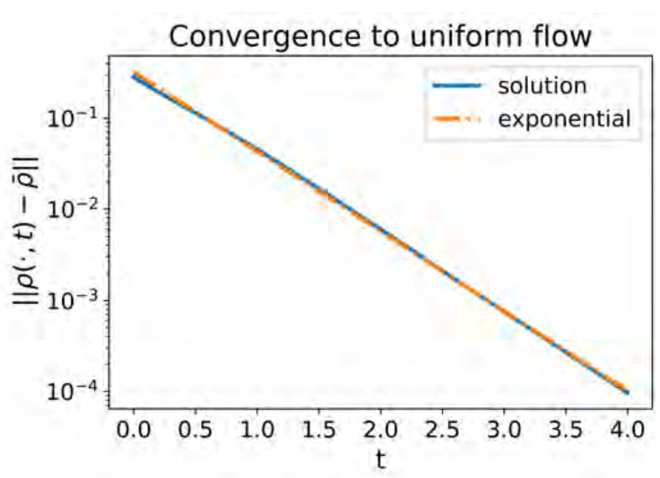
- Nonlocal LWR with $w_\delta(s) = \frac{1}{\delta}$:





- Nonlocal LWR with $w_\delta(s) = \frac{2}{\delta^2}(\delta - s)$:





Global Stability Theorem

Under the following assumptions:

- All vehicles drive on a ring road (periodic boundary condition);
- The velocity function is linear: $U(\rho) = 1 - \rho$;
- The nonlocal kernel $w_\delta(s)$ is C^1 smooth, non-negative, **non-increasing and non-constant**;
- The initial density satisfies $0 < \rho_{\min} \le \rho(x,0) \le \rho_{\max} \le 1$;

and suppose $\rho(x,t)$ is the solution of the nonlocal LWR model, $\bar\rho$ is the average density. Then there exists a constant $\lambda > 0$ that only depends on the nonlocal range δ and nonlocal kernel $w_\delta(s)$, such that:

$$\|\rho(\cdot,t) - \bar\rho\|_{\mathbf{L}^2} \le e^{-\lambda t} \|\rho(\cdot,0) - \bar\rho\|_{\mathbf{L}^2}, \quad \forall t \ge 0.$$

As a corollary, **the traffic density $\rho(\cdot,t)$ very quickly converges to the uniform flow $\bar\rho$ as $t \to \infty$.**

Key Ingredients

- The model can be rewritten as:

$$\partial_t \rho(x,t) + \partial_x \left(\rho(x,t)\left(1 - \rho(x,t)\right)\right) = \nu(\delta)\partial_x \left(\rho(x,t)\mathcal{D}_x^\delta \rho(x,t)\right),$$

where $\mathcal{D}_x^\delta$ is the nonlocal derivative operator:

$$\mathcal{D}_x^\delta \rho(x,t) = \frac{1}{\nu(\delta)} \int_0^\delta \left[\rho(x+s,t) - \rho(x,t)\right] w_\delta(s)\, ds,$$

$$\nu(\delta) = \int_0^\delta s w_\delta(s)\, ds.$$

The nonlocal diffusion $\partial_x \left(\rho(x,t)\mathcal{D}_x^\delta \rho(x,t)\right)$ can dissipate all traffic waves.

- Maximum principle:

$$\rho_{\min} \le \rho(x,t) \le \rho_{\max},$$

at any time $t \ge 0$.

- Define the energy functional as:

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho(x,t) - \bar\rho\right)^2 dx,$$

then:

$$\frac{dE(t)}{dt} = -\nu(\delta)\int_0^1 \rho(x,t)\partial_x \rho(x,t)\mathcal{D}_x^\delta \rho(x,t)\, dx.$$

- Nonlocal Poincare inequality:

$$\int_0^1 \partial_x \rho(x,t)\mathcal{D}_x^\delta \rho(x,t)\, dx \ge \alpha \int_0^1 \left(\rho(x,t) - \bar\rho\right)^2 dx,$$

α only depends on δ and $w_\delta(s)$.

- Nonlinear nonlocal Poincare inequality:

$$\int_0^1 \rho(x,t)\partial_x \rho(x,t)\mathcal{D}_x^\delta \rho(x,t)\, dx \ge \rho_{\min} \int_0^1 \partial_x \rho(x,t)\mathcal{D}_x^\delta \rho(x,t)\, dx.$$

- Energy estimate:

$$\frac{dE(t)}{dt} \le -\nu(\delta)\alpha\rho_{\min} E(t), \quad \forall t \ge 0.$$

Applying Gronwall's lemma yields exponential decay of $E(t)$.

References

[1] M.J.Lighthill, G.B.Whitham, On kinematic waves II: A theory of traffic flow on long, crowded roads. Proceedings of the Royal Society of London Series, 1955.

[2] P.I.Richards, Shock waves on the highway, Operations Research, 1956.

[3] Goatin P, Scialanga S. Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity. Networks & Heterogeneous Media,2016

Further Information

- Paper submitted: Stability of a Nonlocal Traffic Flow Model for Connected Vehicles, with Q. Du. (https://arxiv.org/abs/2007.13915)
- Research supported in part by NSF DMS-2012562, and ARO MURI Grant W911NF-15-1-0562.
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