A new direct discontinuous Galerkin method with interface correction for compressible Navier-Stokes equations

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Problem

Navier-Stokes equations are given as

$$\frac{\partial \mathsf{Q}}{\partial t} + \nabla \cdot \mathsf{F}_c(\mathsf{Q}) = \nabla \cdot \mathsf{F}_v(\mathsf{Q}, \nabla \mathsf{Q}), \quad (\mathsf{x}, t) \in \Omega \times (0, T).$$

where $\mathsf{Q} = (\rho, \rho u, \rho v, E)^T$. The viscous flux is defined as

$$\mathsf{F}_v(\mathsf{Q}, \nabla \mathsf{Q}) = \begin{pmatrix} 0 & 0 \\ \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ u\tau_{11} + v\tau_{21} + \frac{\mu\gamma}{Pr} e_x & u\tau_{12} + v\tau_{22} + \frac{\mu\gamma}{Pr} e_y \end{pmatrix}$$

The viscous stress tensor is

$$\boldsymbol{\tau} = \mu \begin{pmatrix} \frac{4}{3}u_x - \frac{2}{3}v_y & u_y + v_x \\ u_y + v_x & -\frac{2}{3}u_x + \frac{4}{3}v_y \end{pmatrix}.$$

The DG scheme for the NS equations is given as

$$\int_K \mathsf{Q}_t \, v \, dxdy = \int_K (\mathsf{F}_c(\mathsf{Q}) - \mathsf{F}_v(\mathsf{Q}, \nabla \mathsf{Q})) \cdot \nabla v \, dxdy - \int_{\partial K} \left(\widehat{\mathsf{F}_c}(\mathsf{Q}) - \widehat{\mathsf{F}_v}(\mathsf{Q}, \nabla \mathsf{Q})\right) \cdot v \, \mathsf{n} \, ds - \int_{\partial K} \mathsf{I}_{corr}(\mathsf{Q}, v) \cdot \mathsf{n} \, ds.$$

The original direct DG scheme fails to formulate $\widehat{\mathsf{F}_v}(\mathsf{Q}, \nabla \mathsf{Q})$ and $\mathsf{I}_{corr}(\mathsf{Q}, v)$.

Our objective is to reformulate the direct DG method with interface correction to compute the viscous numerical flux and interface correction.

Contributions

- Easy-to-implement new numerical flux and interface correction definitions.
- A nonlinear diffusion model based on sum of multiple individual diffusion processes.
- An algorithm that is extensible to more general conservation laws such as turbulence models and turbulent combustion.

The direct DG method for scalar nonlinear diffusion equations

A single nonlinear diffusion process can be modeled by

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathsf{A}(u) \nabla u), \quad (\mathsf{x}, t) \in \Omega \times (0, T),$$

The weak form is given by

$$\int_K u_t v \, dxdy = \int_{\partial K} \widehat{\mathsf{A}(u)\nabla u} \cdot \mathsf{n} \, v \, ds - \int_K \mathsf{A}(u) \nabla u \cdot \nabla v \, dxdy$$

We define the interface jump and average operators:

$$[\![u]\!] = u^+ - u^-, \quad \{\!\{u\}\!\} = \frac{u^+ + u^-}{2}.$$

The original direct DG numerical flux formula:

$$\widehat{a_{ij}(u)u_{x_j}} = \frac{\beta_0}{h}[\![b_{ij}(u)]\!] n_j + \{\!\{b_{ij}(u)_{x_j}\}\!\} + \beta_1 h [\![b_{ij}(u)_{x_1 x_j} n_1 + b_{ij}(u)_{x_2 x_j} n_2]\!].$$

where $b_{ij}(u) = \int^u a_{ij}(s) ds$ might not be explicitly available.

Our new method for scalar diffusion equations

We consider the adjoint-property of the inner product:

$$\mathsf{A}(u)\nabla u \cdot \mathsf{n} = \nabla u \cdot \mathsf{A}(u)^T \mathsf{n} = \nabla u \cdot \boldsymbol{\xi}(u)$$

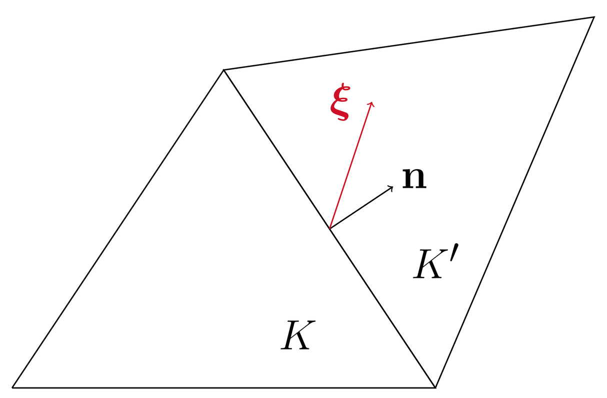
where $\boldsymbol{\xi}(u) = \mathsf{A}(u)^T \mathsf{n}$. On the discrete level, we redefine the numerical flux as:

$$\widehat{\mathsf{A}(u)\nabla u} \cdot \mathsf{n} = \widehat{\nabla u} \cdot \boldsymbol{\xi}(\{\!\{u\}\!\}) \quad \text{with} \quad \boldsymbol{\xi}(\{\!\{u\}\!\}) = \mathsf{A}(\{\!\{u\}\!\})^T \mathsf{n}$$

and compute the numerical flux only for the solution gradient.

$$\widehat{u_x} = \beta_0 \frac{[\![u]\!]}{h} n_1 + \{\!\{u_x\}\!\} + \beta_1 h [\![u_{xx} n_1 + u_{yx} n_2]\!],$$

$$\widehat{u_y} = \beta_0 \frac{[\![u]\!]}{h} n_2 + \{\!\{u_y\}\!\} + \beta_1 h [\![u_{xy} n_1 + u_{yy} n_2]\!].$$



The new direct DG method with interface correction for NS equations

The viscous flux $\widehat{\mathsf{F}_v}(\mathsf{Q}, \nabla \mathsf{Q}) \cdot \mathsf{n}$ can be written as a combination of *multiple individual diffusion processes*.

We consider the x-momentum equation:

$$\mathsf{F}_v^{(2)} \cdot \mathsf{n} = \begin{pmatrix} \tau_{11} \\ \tau_{12} \end{pmatrix} \cdot \mathsf{n} = \mu \begin{pmatrix} \frac{4}{3}u_x - \frac{2}{3}v_y \\ u_y + v_x \end{pmatrix} \cdot \mathsf{n} = \mathsf{A}^{(21)}(\mathsf{Q}) \nabla \rho \cdot \mathsf{n} + \mathsf{A}^{(22)}(\mathsf{Q}) \nabla (\rho u) \cdot \mathsf{n} + \mathsf{A}^{(23)}(\mathsf{Q}) \nabla (\rho v) \cdot \mathsf{n}$$

where the diffusion matrices are given by

$$\mathsf{A}^{(21)}(\mathsf{Q}) = -\frac{\mu}{\rho} \begin{pmatrix} \frac{4}{3}u & -\frac{2}{3}v \\ v & u \end{pmatrix}, \quad \mathsf{A}^{(22)}(\mathsf{Q}) = \frac{\mu}{\rho} \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{A}^{(23)}(\mathsf{Q}) = \frac{\mu}{\rho} \begin{pmatrix} 0 & -\frac{2}{3} \\ 1 & 0 \end{pmatrix}.$$

Defining $\boldsymbol{\xi}^{(2m)}(\mathsf{Q}) = \mathsf{A}^{(2m)}(\mathsf{Q})^T \mathsf{n}$ for $m = 1, 2, 3$, we can write

$$\mathsf{F}_v^{(2)} \cdot \mathsf{n} = \mu \begin{pmatrix} \frac{4}{3}u_x - \frac{2}{3}v_y \\ u_y + v_x \end{pmatrix} \cdot \mathsf{n} = \nabla \rho \cdot \boldsymbol{\xi}^{(21)}(\mathsf{Q}) + \nabla(\rho u) \cdot \boldsymbol{\xi}^{(22)}(\mathsf{Q}) + \nabla(\rho v) \cdot \boldsymbol{\xi}^{(23)}(\mathsf{Q}).$$

This holds true on the continuous level.

At the discrete level, the numerical flux for the viscous term is then defined as:

$$\widehat{\mathsf{F}_v}(\mathsf{Q}, \nabla \mathsf{Q}) \cdot \mathsf{n} = \begin{pmatrix} 0 \\ \widehat{\nabla \rho} \cdot \boldsymbol{\xi}^{(21)} + \widehat{\nabla(\rho u)} \cdot \boldsymbol{\xi}^{(22)} + \widehat{\nabla(\rho v)} \cdot \boldsymbol{\xi}^{(23)} \\ \widehat{\nabla \rho} \cdot \boldsymbol{\xi}^{(31)} + \widehat{\nabla(\rho u)} \cdot \boldsymbol{\xi}^{(32)} + \widehat{\nabla(\rho v)} \cdot \boldsymbol{\xi}^{(33)} \\ \widehat{\nabla \rho} \cdot \boldsymbol{\xi}^{(41)} + \widehat{\nabla(\rho u)} \cdot \boldsymbol{\xi}^{(42)} + \widehat{\nabla(\rho v)} \cdot \boldsymbol{\xi}^{(43)} + \widehat{\nabla E} \cdot \boldsymbol{\xi}^{(44)} \end{pmatrix}.$$

The interface correction term can be derived similarly:

$$\mathsf{I}_{corr}(\mathsf{Q}, v) \cdot \mathsf{n} = \frac{1}{2} \begin{pmatrix} 0 \\ \left([\![\rho]\!]\boldsymbol{\xi}^{(21)} + [\![(\rho u)]\!]\boldsymbol{\xi}^{(22)} + [\![(\rho v)]\!]\boldsymbol{\xi}^{(23)}\right) \cdot \nabla v \\ \left([\![\rho]\!]\boldsymbol{\xi}^{(31)} + [\![(\rho u)]\!]\boldsymbol{\xi}^{(32)} + [\![(\rho v)]\!]\boldsymbol{\xi}^{(33)}\right) \cdot \nabla v \\ \left([\![\rho]\!]\boldsymbol{\xi}^{(41)} + [\![(\rho u)]\!]\boldsymbol{\xi}^{(42)} + [\![(\rho v)]\!]\boldsymbol{\xi}^{(43)} + [\![E]\!]\boldsymbol{\xi}^{(44)}\right) \cdot \nabla v \end{pmatrix}.$$

where $\boldsymbol{\xi}^{(lm)} = \boldsymbol{\xi}^{(lm)}(\{\!\{\mathsf{Q}\}\!\}) = \mathsf{A}^{(lm)}(\{\!\{\mathsf{Q}\}\!\})^T \mathsf{n}$ for $l = 2, 3, 4, \quad m = 1, \cdots, 4$

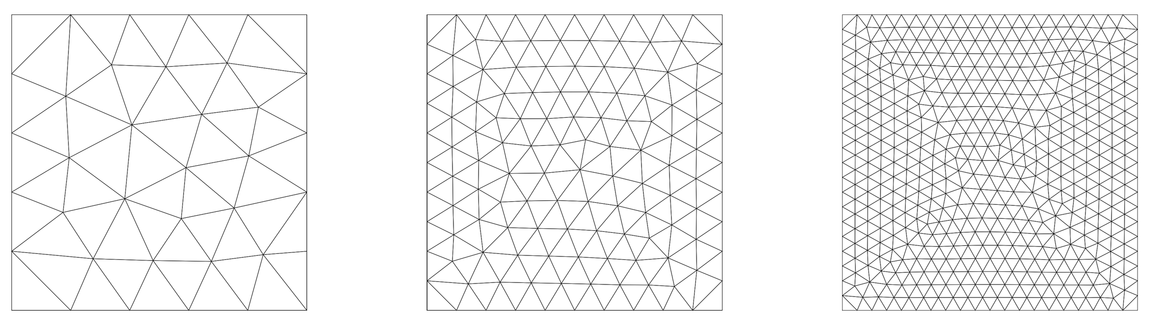
A pressure pulse in a periodic domain

We consider a square domain $\Omega = [0, 1] \times [0, 1]$ with the initial conditions:

$$\rho(x, y, 0) = 1, \; u(x, y, 0) = v(x, y, 0) = 0,$$

$$E(x, y, 0) = \frac{12}{\gamma - 1} + \frac{1}{2}\exp\left(-\left(\cos(\pi x)^2 + \cos(\pi y)^2\right)\right).$$

Error analysis has been conducted on a set of triangular meshes:



We obtain $(k+1)th$ order convergence.

L_2 errors and orders for (ρu)

	h	$h/2$	Order	$h/4$	Order	$h/8$	Order
$k=3$	6.08E-5	2.57E-6	4.56	1.60E-7	4.00	9.28E-9	4.11
$k=4$	1.05E-5	2.62E-7	5.33	8.53E-9	4.94	2.76E-10	4.95

L_2 errors and orders for E

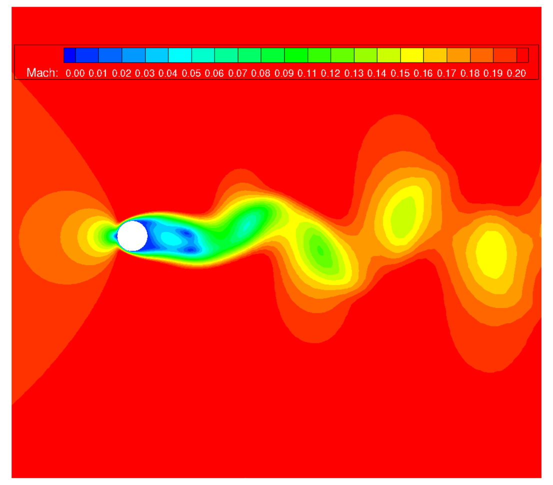
	h	$h/2$	Order	$h/4$	Order	$h/8$	Order
$k=3$	7.71E-4	2.66E-5	4.86	1.65E-6	4.02	9.88E-8	4.06
$k=4$	1.16E-4	2.91E-6	5.31	7.91E-8	5.20	3.12E-9	4.66

Unsteady flow over a cylinder

The new direct DG method accurately calculates the physical quantities:

- $M = 0.2$ and $Re = 75$
- St is Strouhal number
- C_l is lift coefficient
- C_d is drag coefficient

	C_d	C_l	$St = fD/U$
$k=3$	1.396 ± 0.00342	± 0.211	0.150
$k=4$	1.396 ± 0.00351	± 0.211	0.150
Canuto and Taira	1.390 ± 0.00310	± 0.212	0.150



Future Work

- Stability and adjoint consistency analysis.
- Positivity-preserving limiter.
- Extension to turbulence simulations.

References

[1] Daniel Canuto and Kunihiko Taira.
Two-dimensional compressible viscous flow around a circular cylinder.
Journal of fluid mechanics, 785:349–371, 2015.

[2] Mustafa E Danis and Jue Yan.
A new direct discontinuous galerkin method with interface correction for two-dimensional compressible navier-stokes equations.
arXiv preprint arXiv:2104.09767, 2021.

[3] Hailiang Liu and Jue Yan.
The direct discontinuous galerkin (ddg) method for diffusion with interface corrections.
Communications in Computational Physics, 8(3):541, 2010.