We are interested in equations of the form:
\[ \partial_t u + \partial_x f(u) = 0, \]
where \( u : [0, \infty) \times \mathbb{R} \to V \subset \mathbb{R}^n \) is the state and \( f : V \to \mathbb{R}^n \) is the associated flux.

### Entropy Solutions:
We say \( \eta, q : V \to \mathbb{R} \) is an entropy-entropy flux pair for the system \( (\eta) \) if
\[ \forall q = \nabla \eta f, \]
We say \( u \) is an entropy solution to \( (\eta) \) if \( u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \) satisfies \( (\eta) \) in the sense of distributions and verifies
\[ \partial_t \eta(u) + \partial_x q(u) \leq 0, \]
for each entropy-entropy flux pair.

### Typical Fluxes:
- **Isentropic Euler:** Let \( \gamma > 1 \).
\[ \rho_t + (\rho u)_x = 0 \]
(3)
\[ \rho u_t + (\rho u^2 + p)_x = 0 \]
(4)
- **Full Euler:** Let \( E = \frac{1}{2} u^2 + e \) and fix a pressure law \( P(\rho, e) \).
\[ \rho_t + (\rho u)_x = 0 \]
(5)
\[ \rho u_t + (\rho u^2 + P(\rho, e))_x = 0 \]
(6)
\[ (\rho E)_t + (\rho uE + uP(\rho, e))_x = 0 \]
(7)

### Classical Results:
- **Existence of global-in-time entropy solutions** is known in several scenarios using the vanishing viscosity method, compensated compactness, or convergence of numerical schemes [Glimm, 1965].
- **For scalar, \( V \subset \mathbb{R} \), in [Kruzkov, 1970] and \( u, v \) entropy solutions with initial data \( u_0, v_0 \).
\[ \|u(t) - v(t)\|_1 \leq \|u_0 - v_0\|_1. \]
(8)
- **For \( V \subset \mathbb{R}^n \), in [Diperna,1979] and [Dafermos, 1979], stability of Lipschitz solutions among all entropy solutions \( 0. \)
- **Separately, \( V \subset \mathbb{R}^n \), if \( u_0 \) small BV, in [Bressan and Lewicka, 2000] there is at most one solution \( u \) which has bounded variation along space-like curves and initial data \( u_0 \).**

### The Relative Entropy Method:
For \( \eta \) a strictly convex entropy of \( (\eta) \) with entropy flux \( q \), the associated relative entropy and entropy flux functionals are
\[ \eta(u|v) = \eta(u) - \eta(v) - \int_v^u \eta' \ dX, \]
\[ \eta(u|v) = \eta(u) - \eta(v) - \int_v^u \eta' \ dX, \]
(9)
(10)
- **If \( u \) is an entropy solution to \( (\eta) \), \( u \) satisfies**
\[ \partial_t \eta(u|v) + \partial_x q(u|v) \leq 0 \text{ for } \forall v \in V. \]
(11)
- **Taylor expansion using strict convexity of \( \eta \) gives \( \eta(u|v) \approx |u - v|^2 \).**
- **Introduced by Dafermos and DiPerna to show stability of Lipschitz solutions among entropy solutions by taking \( u \) a weak solution and \( v \) a Lipschitz solution.**

The method is \( L^1 \)-based, which is bad for stability of discontinuous solutions. Consider the inviscid \( n \)d Burger’s equation,
\[ \partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0. \]
(12)

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**Figure:** We track the \( L^2 \) norm difference of a shock and a perturbation without a shift. We use the Lax-Friedrichs scheme with \( \Delta t = 0.001, \Delta x = 0.002 \).

The \( \alpha \)-Contraction Method:
- **The \( \alpha \)-contraction property for \( u(t, x) \) an entropic solution and \( (u_1, u_2, \sigma_\alpha) \) an entropic \( \alpha \)-shock holds if there are weights \( a_1, a_2 \in \mathbb{R} \) and a Lipschitz shift \( t \to h(t) \) such that**
\[ E(t) = \int_{(0,1)} a_\alpha(u_1) \ dx + \int_{(0,1)} a_\alpha(u_2) \ dx \leq E(0). \]
(13)
- **The \( \alpha \)-contraction property holds with weights \( a_1 = a_2 = 1 \) for many scalar problems, but the weights are necessary for the majority of systems [Leger, 2011]. [Serre and Vasseur, 2014].**
- **But with weights \( \alpha \)-contraction property holds for a large family of \( n \times n \) systems with small \( a_1, a_2 \) [Kang and Vasseur, 2015].**
- **Main question:** For \( S(t, x) \) the solution corresponding to \( (u_1, u_2, \sigma_{\alpha_2}) \), how far is \( E(t) \) from \( \|u(t, \cdot + h(t)) - S(t, \cdot)\|_2 \)?

**Definition: Strong Trace:**
We say a function \( u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \) satisfies the strong trace property if for each \( t \to X(t) \) a Lipschitz curve, there are \( u^k \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \) such that for any \( T \to 0 \),
\[ \lim_{T \to 0} \int_0^T \text{ess sup} \ |u(t, X(t) + y) - u^k(t, dt) = 0. \]
(14)

Suppose \( \eta \) corresponds to a class of fluxes containing the isentropic and full Euler (with ideal gas law equation of state) systems and \( u \) is an entropy solution satisfying the strong trace property.

**Theorem 1:** G, Vasseur [2020]
Then, for any \( d \in Y \) there are \( C, \alpha, \Delta > 0 \) such that for any \( (u_1, u_2, \sigma_{\alpha_2}) \) a \( \alpha \)-shock with \( u_1, u_2 \in B_{(d)} \), there are \( a_1, a_2 > 0 \) and \( t \to h(t) \) Lipschitz such that for almost every \( t \),
\[ E(t) = \int_{(0,1)} a_\alpha(u_1) \ dx + \int_{(0,1)} a_\alpha(u_2) \ dx \leq E(0) \]
(15)
\[ 1 + C' |u_1 - u_2| \leq a_\alpha \leq 1 + 2C' |u_1 - u_2| \]
(16)
\[ -\alpha_1 \leq h(t) \leq \alpha_2 \leq \text{inf } \lambda_2(u) \]
(17)

Suppose \( \alpha \) belongs to a large class of fluxes with \( V \subset \mathbb{R}^2 \), which includes the flux from isentropic Euler.

**Theorem 1.2:** Chen, Krupa, Vasseur [2020]
There is an \( \varepsilon > 0 \) such that for \( \|u_1|_{BV} < \varepsilon \), if \( u(t, x) \) and \( v(t, x) \) are entropy solutions to \( (\eta) \) with initial data \( u_0, v_0 \), \( \|u_0 - v_0\|_{BV} < \varepsilon \), and \( v \) satisfying the strong trace property, then \( u = v \).

**Implications of Theorem 1.2**
Theorem 1.2 implies that the Tame Oscillation condition and the Bounded Variations along Space-Like Curves condition of Bressan et al. are not needed for the stability of small \( BV \) entropy solutions, at least in the \( 2 \times 2 \) case.

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May 19, 2021