Sharp a-Contraction Estimates for Small Extremal Shocks

Based on joint work with Sam Krupa and Alexis Vasseur

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We are interested in equations of the form:

$$\partial_t u + \partial_x f(u) = 0, \tag{1}$$

where $u : [0,\infty)\times\mathbb{R}\to\mathcal{V}\subset\mathbb{R}^n$ is the state and $f:\mathcal{V}\to\mathbb{R}^n$ is the associated flux.

Entropy Solutions:

We say $\eta, q : \mathcal{V}\to\mathbb{R}$ is an entropy-entropy flux pair for the system (1) if $\nabla q = \nabla\eta\nabla f$. We say u is an entropy solution to (1) if $u\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ satisfies (1) in the sense of distributions and verifies

$$\partial_t\eta(u) + \partial_x q(u) \le 0, \tag{2}$$

for each entropy-entropy flux pair.

Typical Fluxes:

- Isentropic Euler: Let $\gamma > 1$.

$$\rho_t + (\rho u)_x = 0 \tag{3}$$
$$(\rho u)_t + (\rho u^2 + \rho^\gamma)_x = 0 \tag{4}$$

- Full Euler: Let $E = \frac{1}{2}u^2 + e$ and fix a pressure law $P(\rho, e)$.

$$\rho_t + (u\rho)_x = 0 \tag{5}$$
$$(\rho u)_t + (\rho u^2 + P(\rho,e))_x = 0 \tag{6}$$
$$(\rho E)_t + (\rho u E + uP(\rho,e))_x = 0 \tag{7}$$

Classical Results:

- Existence of global-in-time entropy solutions is known in several scenarios using the vanishing viscosity method, compensated compactness, or convergence of numerical schemes [Glimm, 1965].
- For scalar, $\mathcal{V}\subset\mathbb{R}$, in [Kruzkov, 1970] and u, v entropy solutions with initial data u_0, v_0,

$$\|u(t)-v(t)\|_{L^1_x} \le \|u_0 - v_0\|_{L^1}. \tag{8}$$

- For $\mathcal{V}\subset\mathbb{R}^n$, in [Diperna,1979] and [Dafermos, 1979], stability of Lipschitz solutions among all entropy solutions .
- Separately, $\mathcal{V}\subset\mathbb{R}^n$, if u_0 small BV, in [Bressan and Lewicka, 2000] there is at most one solution u which has bounded variation along space-like curves and initial data u_0.

The Relative Entropy Method:

For η a strictly convex entropy of (1) with entropy flux q, the associated relative entropy and entropy flux functionals are

$$\eta(u|v) = \eta(u) - \eta(v) - \nabla\eta(u)\cdot[u-v] \tag{9}$$
$$q(u;v) = q(u) - q(v) - \nabla\eta(u)\cdot[f(u)-f(v)]. \tag{10}$$

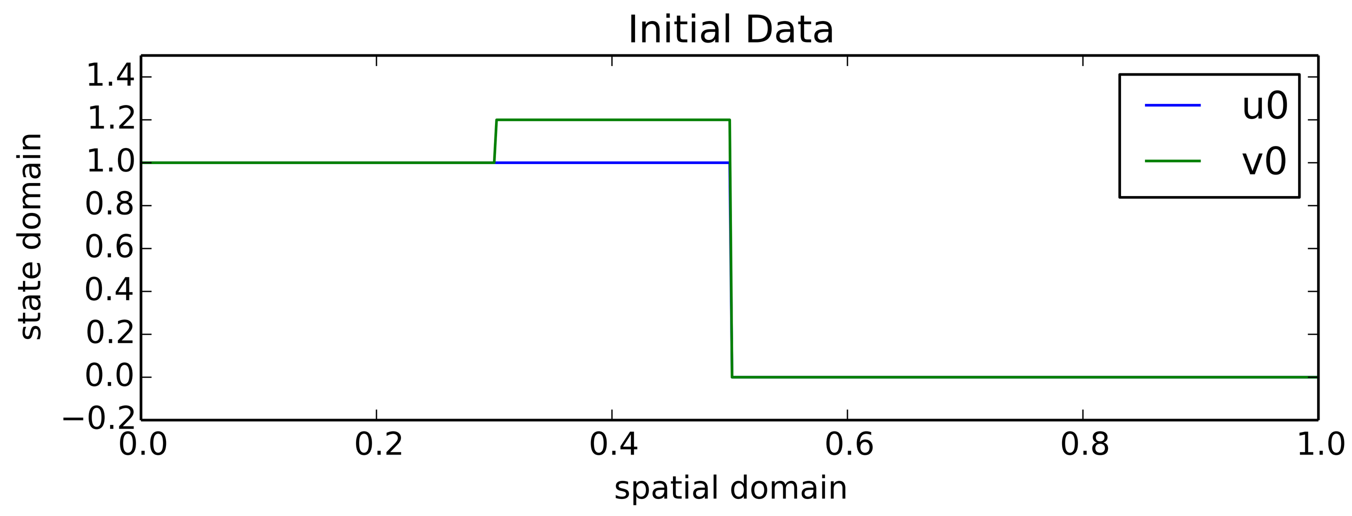
- If u is an entropy solution to (1), u satisfies

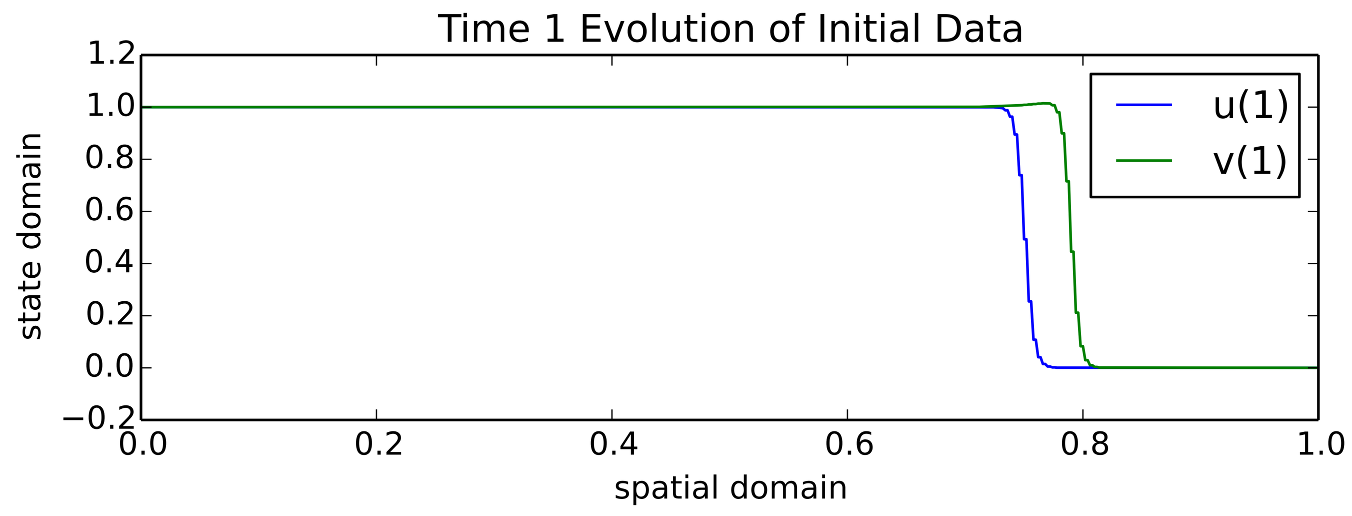
$$\partial_t\eta(u|v) + \partial_x q(u;v) \le 0 \quad \text{for } v\in\mathcal{V}. \tag{11}$$

- Taylor expansion using strict convexity of η gives $\eta(u|v)\sim|u-v|^2$.
- Introduced by Dafermos and DiPerna to show stability of Lipschitz solutions among entropy solutions by taking u a weak solution and v a Lipschitz solution.

The method is L^2-based, which is bad for stability of discontinuous solutions. Consider the inviscid 1d Burger's equation,

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0\,. \tag{12}$$





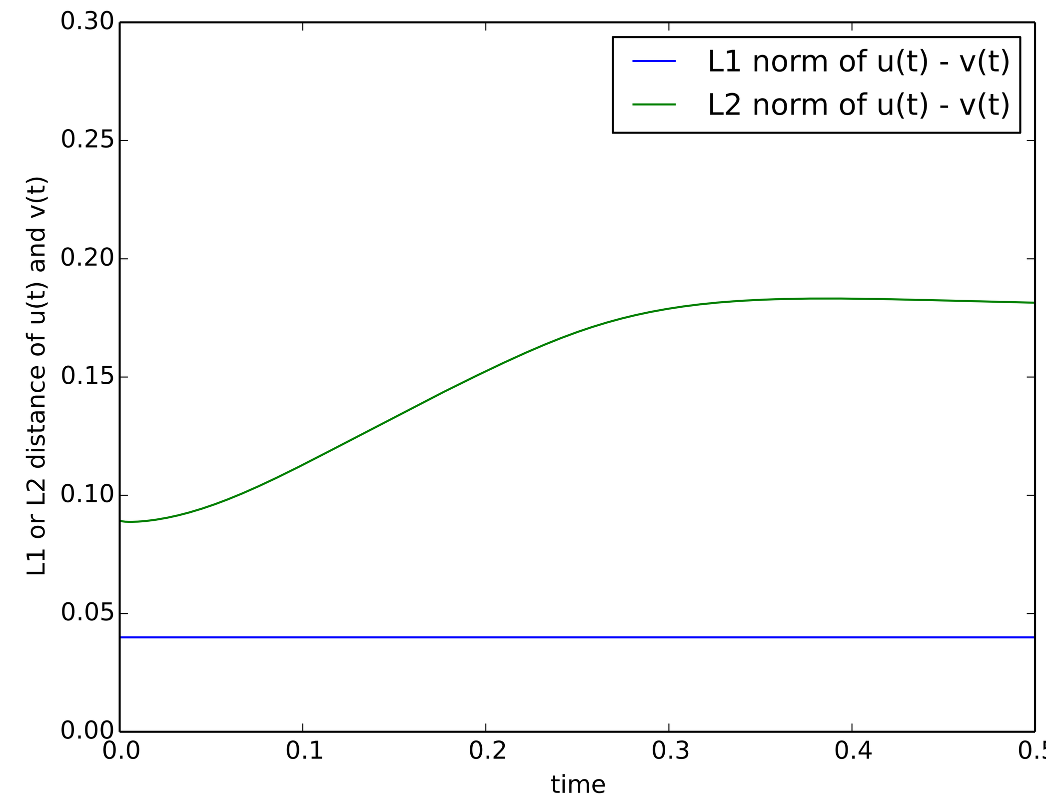


Figure: We track the L^2 norm difference of a shock and a perturbation without a shift. We use the Lax-Friedrichs scheme with $\Delta t = 0.001$, $\Delta x = 0.002$

The a-Contraction Method:

- The a-contraction property for $u(t,x)$ an entropic solution and (u_L, u_R, σ_{LR}) an entropic 1-shock holds if there are weights $a_1, a_2\in\mathbb{R}$ and a Lipschitz shift $t\mapsto h(t)$ such that

$$E(t) = \int_{-\infty}^{h(t)} a_1\eta(u|u_L)\,dx + \int_{h(t)}^{\infty} a_2\eta(u|u_R)\,dx \le E(0). \tag{13}$$

- The a-contraction property holds with weights $a_1 = a_2 = 1$ for many scalar problems, but the weights are necessary for the majority of systems [Leger, 2011], [Serre and Vasseur, 2014].
- But with weights, a-contraction property holds for a large family of $n\times n$ systems with small a_1, a_2 [Kang and Vasseur, 2015].
- Main question: For $S(t,x)$ the solution corresponding to (u_L, u_R, σ_{LR}), how far is $E(t)$ from $\|u(t,\cdot+h(t)) - S(t,\cdot)\|_{L^2}^2$?

Definition: Strong Trace

We say a function $u\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ satisfies the strong trace property if for each $t\mapsto X(t)$ a Lipschitz curve, there are $u^\pm\in L^\infty(\mathbb{R}^+)$ such that for any $T>0$,

$$\lim_{k\to\infty}\int_0^T \operatorname*{ess\,sup}_{(0,1/k)} |u(t, X(t)\pm y) - u^\pm(t)|\,dt = 0\,. \tag{14}$$

Suppose f corresponds to a class of fluxes containing the isentropic and full Euler (with ideal gas law equation of state) systems and u is an entropy solution satisfying the strong trace property.

[Theorem 1; G., Krupa, Vasseur 2020]

Then, for any $d\in\mathcal{V}$ there are $\varepsilon,\ C',\ \alpha_1,\ \alpha_2 > 0$ such that for any (u_L, u_R, σ_{LR}) a 1-shock with $u_L, u_R\in B_\varepsilon(d)$, there are $a_1,\ a_2 > 0$ and $t\mapsto h(t)$ Lipschitz such that for almost every t,

$$E(t) = \int_{-\infty}^{h(t)} a_1\eta(u|u_L)\,dx + \int_{h(t)}^{\infty} a_2\eta(u|u_R)\,dx \le E(0) \tag{15}$$
$$1 + C'|u_L - u_R| \le \frac{a_1}{a_2} \le 1 + 2C'|u_L - u_R| \tag{16}$$
$$-\alpha_1 \le \dot{h}(t) \le \alpha_2 < \inf_{u\in B_\varepsilon(d)} \lambda_2(u) \tag{17}$$

Suppose f belongs to a large class of fluxes with $\mathcal{V}\subset\mathbb{R}^2$, which includes the flux from isentropic Euler.

[Theorem 1.3; Chen, Krupa, Vasseur 2020]

There is an $\varepsilon > 0$ such that for $\|u_0\|_{BV} < \varepsilon$, if $u(t,x)$ and $v(t,x)$ are entropy solutions to (1) with initial data u_0, $\|u\|_{L_t^\infty BV_x} < \varepsilon$, and v satisfying the strong trace property, then $u = v$.

Implications of Theorem 1.3

Theorem 1.3 implies that the Tame Oscillation condition and the Bounded Variations along Space-like Curves condition of Bressan et al. are not needed for the stability of small BV entropy solutions, at least in the 2×2 case.

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