Energy, Entropy, and Dissipative Dynamics

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Entropy Methods for Gas Dynamics on Networks

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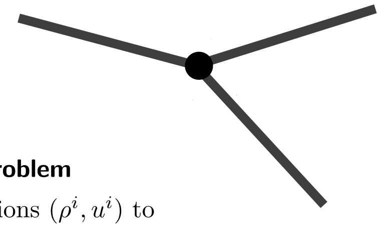

Isentropic gas dynamics

We consider weak entropy solutions to

$$\begin{cases} \partial_t \rho^i + \partial_x (\rho u)^i & = 0 \\ \partial_t (\rho u)^i + \partial_x (\rho u^2 + \kappa \rho^\gamma)^i & = 0 \end{cases} \quad t, x > 0,$$

with density $\rho^i \geq 0$, flow velocity $u \in \mathbb{R}$, $\kappa > 0$, $1 < \gamma < 3$ and $i = 1, \dots, n$.

A kinetic BGK model [Bouchut, 1999]

$$\partial_t f^i + \xi \, \partial_x f^i = \frac{1}{\epsilon} (M[f^i] - f^i), \quad t, x > 0,$$

for $f^i = f^i(t, x, \xi) \in \mathbb{R}^2$, $f_0^i \geq 0$ and **Maxwellian** $M[f] = M(\rho_f, u_f, \xi)$ with moments $(\rho_f, \rho_f u_f) = \int_{\mathbb{R}} f(\xi) \, \mathrm{d}\xi$.

Maximum energy dissipation problem

We select a coupling condition

$$f^i(t, x = 0, \xi) = \Psi^i[f^j(t, x = 0, \mathbb{R}^-), 1 \leq j \leq n](\xi)$$

for a.e. $t > 0, \xi > 0$, by solving the maximum energy dissipation problem

$$\inf \sum_{i=1}^d A^i \int_0^\infty \xi \, H(\Psi^i(\xi), \xi) \, \mathrm{d}\xi$$

$$\text{s.t.} \sum_{i=1}^d A^i \left[\int_0^\infty \xi \, \Psi_0^i(\xi) \, \mathrm{d}\xi + \int_{-\infty}^0 \xi \, g_0^i(\xi) \, \mathrm{d}\xi \right] = 0$$

for given data $g^i(\xi)$, $\xi < 0$, where H denotes the kinetic energy to the BGK model. The solution is

$$\Psi^i(\xi) = M(\rho_*, u_* = 0, \xi) \quad \text{for } \xi > 0.$$

for suitable $\rho_* \geq 0$. Solutions to the kinetic BGK model on networks exist if initial total mass and energy are finite [2].

Interior relaxation

By **compensated compactness**, we obtain strong convergence towards weak entropy solutions to the system of isentropic gas dynamics [2].

A new coupling condition

The kinetic approach and assuming $\rho_*^\epsilon \to \rho_*$ in L^1_{loc} leads to the following definition of the generalized Riemann problem. Solutions exist and are unique for all initial data [3], not necessarily being subsonic.

Generalized Riemann problem

1. Self-similar Lax solutions (ρ^i, u^i) to

$$(\rho^i, \rho^i u^i)(t = 0, x) = \begin{cases} (\rho_*, 0) & \text{if } x < 0, \\ (\rho_0^i, \rho_0^i u_0^i) & \text{if } x > 0. \end{cases}$$

2. The artificial density $\rho_* \geq 0$ is chosen such that

$$\sum_{i=1}^d A^i (\overline{\rho u})^i (t, 0+) = 0.$$

3. We restrict $(\rho^i, \rho^i u^i)$ to $x > 0$.

Comparison of coupling conditions

Conservation of total mass

$$\sum_{i=1}^d A^i (\rho u)^i (t, 0+) = 0.$$

Equal pressure [Banda, Herty, Klar, 2006]:

$$(\rho^\gamma)^i (t, 0+) = \mathcal{H}(t) \quad \text{for all } i.$$

Equal momentum flux [Colombo, Garavello, 2006]:

$$(\rho u^2 + \kappa \rho^\gamma)^i (t, 0+) = \mathcal{H}(t) \quad \text{for all } i.$$

Equal stagnation enthalpy [Reigstad, 2015]:

$$(\frac{u^2}{2} + \frac{\gamma \kappa}{\gamma - 1} \rho^{\gamma - 1})^i (t, 0+) = \mathcal{H}(t) \quad \text{for all } i.$$

A numerical example

Set $n = 3$, $\gamma = 2$, $\kappa = 5$, $A^i = 1$ with initial data

$$\begin{cases} \rho_{0,i} = 1 & i = 1, 2, 3, \\ u_{0,1} = -1, & u_{0,2} = u_{0,3} = 1/2 \end{cases}$$

The new coupling condition leads to physically meaningful waves and dissipates energy at the junction:

pipe	new condition	pressure	momentum flux	stagnation enthalpy
1	S	NW	R	R
2	R	NW	S	S
3	R	NW	S	S
energy dissipation	+	0	+/-	0

NW=no wave, S=shock, R=rarefaction wave

References

[1] F. Berthelin, and F. Bouchut. Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics, *Asymptot. Anal.*, 31(2):153–176, 2002.

[2] Y. H., Kinetic relaxation to entropy based coupling conditions for isentropic flow on networks, *J. Differential Equations*, 269(2):1192–1225, 2020.

[3] Y. H., M. Herty, and M. Westdickenberg, New coupling conditions for isentropic flow on networks, *Netw. Heterog. Media*, 15(4):605–631, 2020.

[4] G. A. Reigstad. Existence and uniqueness of solutions to the generalized Riemann problem for isentropic flow, *SIAM J. Appl. Math.*, 75(2):679–702, 2015.

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