Entropy Methods for Gas Dynamics on Networks

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Joint work with Michael Herty, Michael Westdickenberg

Isentropic gas dynamics
We consider weak entropy solutions to
\[\begin{align*}
\partial_t \rho^i + \partial_x (\rho u^i) &= 0, \\
\partial_t (\rho u^i) + \partial_x (\rho u^2 + \kappa \rho^i) &= 0
\end{align*}\]
t, x > 0,
with density \(\rho^i \geq 0\), flow velocity \(u \in \mathbb{R}\), \(\kappa > 0\), 1 < \(\gamma < 3\) and \(i = 1, \ldots, n\).

A kinetic BGK model [Bouchut, 1999]
\[\begin{align*}
\partial_t f^i + \xi \partial_x f^i &= \frac{1}{\epsilon} (M[f^i] - f^i), \\
\partial_t f^i + \partial_x f^i &= \frac{1}{\epsilon} (M[f^i] - f^i),
\end{align*}\]
t, x > 0,
for \(f^i = f^i(t, x, \xi) \in \mathbb{R}^2, f_0^i \geq 0\) and Maxwellian \(M[f] = M(\rho_f, u_f, \xi)\) with moments \((\rho_f, \rho_f u_f) = \int f(\xi) \, d\xi\).

Maximum energy dissipation problem
We select a coupling condition
\[f^i(t, x = 0, \xi) = \Psi^i[f^j(t, x = 0, \mathbb{R}^-), 1 \leq j \leq n](\xi)\]
for a.e. \(t > 0, \xi > 0\), by solving the maximum energy dissipation problem
\[\inf \sum_{i=1}^d A^i \left[ \int_0^\infty c^i \Phi^i(\xi) \, d\xi + \int_{-\infty}^0 c_0^i \Phi^i(\xi) \, d\xi \right] = 0\]
for given data \(g^i(\xi), \xi < 0\), where \(H\) denotes the kinetic energy to the BGK model. The solution is
\[\Phi^i(\xi) = M(\rho^i, u^i = 0, \xi)\]
for \(\xi > 0\).

Interior relaxation
By compensated compactness, we obtain strong convergence towards weak entropy solutions to the system of isentropic gas dynamics [2].

A new coupling condition
The kinetic approach and assuming \(\rho^i \to \rho^*\) in \(L^1_{\text{loc}}\) leads to the following definition of the generalized Riemann problem. Solutions exist and are unique for all initial data [3], not necessarily being subsonic.

Generalized Riemann problem
1. Self-similar Lax solutions \((\rho^1, u^1)\) to
\[\begin{align*}
(\rho^1, u^1)(t = 0, x) &= \left\{ \begin{array}{ll}
(\rho_0^i, 0) & \text{if } x < 0, \\
(\rho_0^i, \rho_0^i u_0^i) & \text{if } x > 0.
\end{array} \right.
\end{align*}\]
2. The artificial density \(\rho^* \geq 0\) is chosen such that
\[\sum_{i=1}^d A^i(\mathcal{P})^i(t, 0+) = 0.\]
3. We restrict \((\rho^1, \rho^1 u^1)\) to \(x > 0\).

Comparison of coupling conditions
Conservation of total mass
\[\sum_{i=1}^d A^i(\rho u)^i(t, 0+) = 0.\]

Equal pressure [Banda, Herty, Klar, 2006]:
\[\rho^i(t, 0+) = \mathcal{H}(t) \quad \text{for all } i.\]

Equal momentum flux [Colombo, Garavello, 2006]:
\[(\rho u^2 + \kappa \rho^i)^i(t, 0+) = \mathcal{H}(t) \quad \text{for all } i.\]

Equal stagnation enthalpy [Reigstad, 2015]:
\[\left( \frac{u^2}{2} + \frac{\gamma \kappa}{\gamma - 1} \rho^{i-1} \right)^i(t, 0+) = \mathcal{H}(t) \quad \text{for all } i.\]

A numerical example
Set \(n = 3, \gamma = 2, \kappa = 5, A^i = 1\) with initial data
\[\begin{align*}
\rho_{0,i} &= 1, & i &= 1, 2, 3, \\
\rho_{0,1} &= 1, & \rho_{0,2} &= \rho_{0,3} = 1/2.
\end{align*}\]

The new coupling condition leads to physically meaningful waves and dissipates energy at the junction:

<table>
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<th>pipe</th>
<th>new condition</th>
<th>pressure</th>
<th>momentum flux</th>
<th>stagnation enthalpy</th>
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<td>NW</td>
<td>R</td>
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<tr>
<td>3</td>
<td>R</td>
<td>NW</td>
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NW=no wave, S=shock, R=rarefaction wave

References

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