A combinatorial description of the totally nonnegative Grassmannian

Kelli Talaska
UC Berkeley

OUTLINE:

I. TNN Grasmannians and the positroid decomposition.

II. A network perspective.

III. Connections.

GRASSMANNIANS

Let $Gr(k,n)$ denote the Grassmannian:
space of all k-diml subspaces of $\mathbb{R}^n$.

Q: How can we represent points in $Gr(k,n)$?

A1: Full rank $k \times n$ matrices (modulo left action by GL_k).

A2: List of $\binom{n}{k}$ Plücker coords (unique up to mult
by common $\neq 0$ scalar)

($= k \times k$ minors of a matrix in A1.)
Δ_J denotes minor w/ cols J.

TNN GRASSMANNIAN

The <u>totally nonnegative Grassmannian</u> $Gr(k,n)_{\geq 0}$ consists of those points in $Gr(k,n)$ whose nonzero Plücker coordinates all have the same sign.
(i.e. can all be simultaneously nonnegative)

Eg: In $Gr(2,4)_{\geq 0}$:

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 1 & 4 & 6 & 12 \end{bmatrix}$$

$\Delta_{12} = 2$ $\qquad$ $\Delta_{23} = 0$

$\Delta_{13} = 3$ $\qquad$ $\Delta_{24} = 40$

$\Delta_{14} = 16$ $\qquad$ $\Delta_{34} = 60$

POSITROID STRATIFICATION

Idea: Specify which $\Delta_J = 0$ and which $\Delta_J \neq 0$.

History: Essentially the TNN part of the matroid stratification studied by GGMS.

Eg: $Gr(2,4)$ has a single Plücker relation:

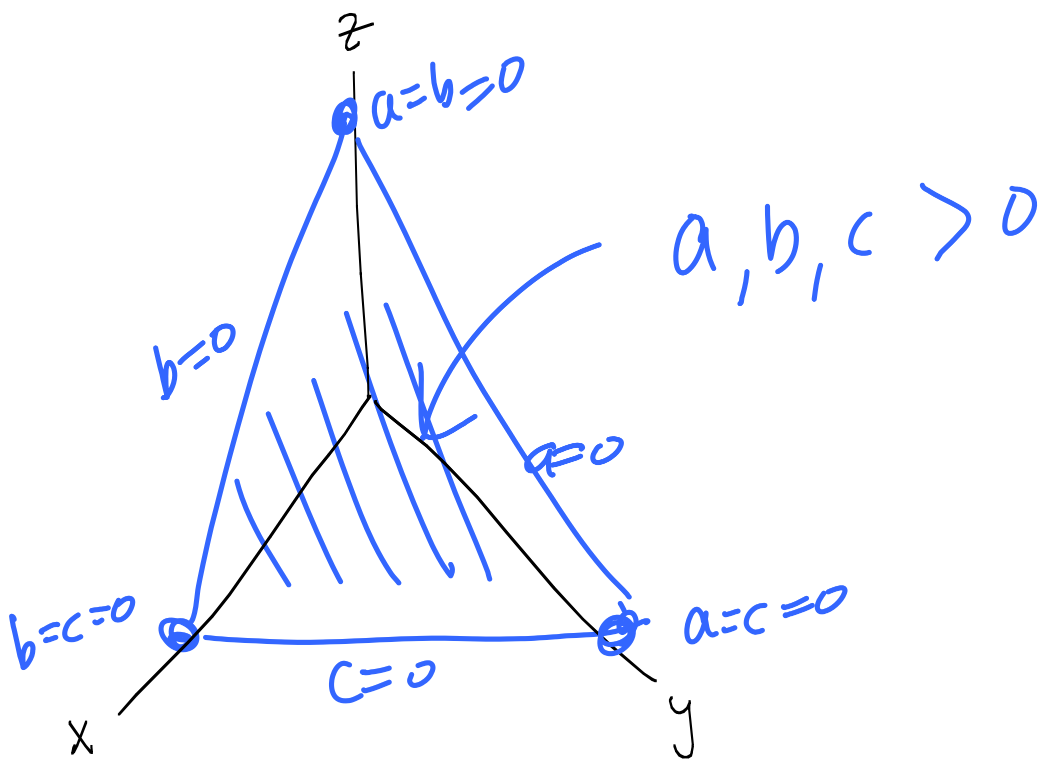
$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

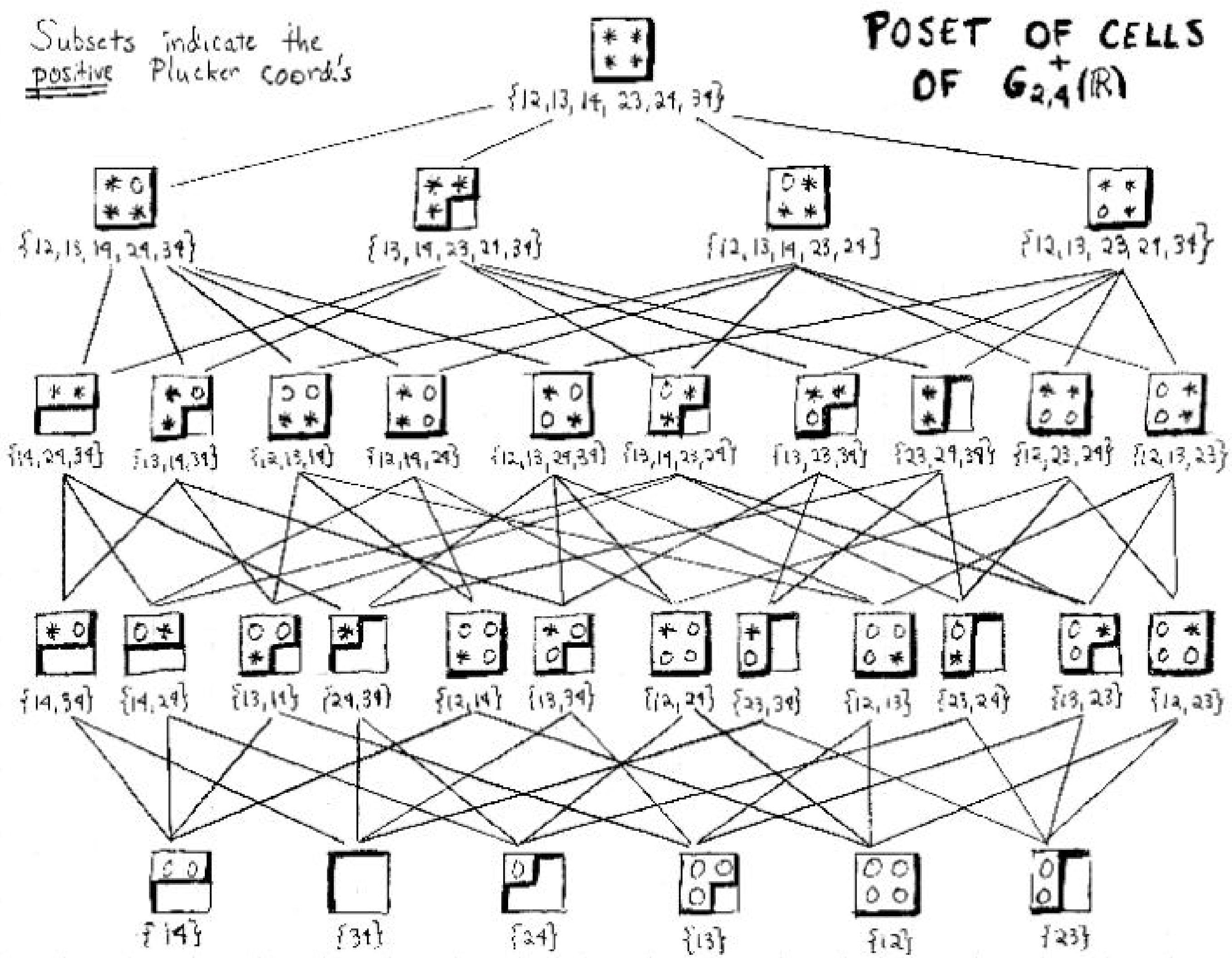
Δ_{13}	Δ_{24}	Δ_{12}	Δ_{34}	Δ_{14}	Δ_{23}	
+	+	0	+	+	+	(nice)
0	*	*	*	*	*	← matroid

positroids

Eg: $Gr(1,3)_{\geq 0}$ matrix $[a \ \ b \ \ c]$

Plücker $\Delta_1 = a$, $\Delta_2 = b$, $\Delta_3 = c$




POSET OF CELLS OF $G^{+}_{2,4}(\mathbb{R})$
Subsets indicate the positive Plucker coord's
{12, 13, 14, 23, 24, 34}
{12, 13, 14, 24, 34}
{13, 14, 23, 24, 34}
{12, 13, 14, 23, 24}
{12, 13, 23, 24, 34}
{14, 24, 34}
{13, 14, 34}
{12, 13, 14}
{12, 14, 24}
{12, 13, 24, 34}
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{13, 34}
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{23, 34}
{12, 13}
{23, 24}
{13, 23}
{12, 23}
{14}
{34}
{24}
{13}
{12}
{23}

Rest of the talk:

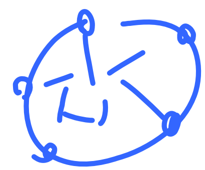
Build a combinatorial understanding of this stratification.

positroid cells $\longleftrightarrow$ Le-diagrams $\longleftrightarrow$ ⅂-graphs

- Use weighted ⅂-graphs to parametrize positroid cells
- In "big" cell, weights can be written in terms of a nice cluster.

POSTNIKOV'S BOUNDARY MEASUREMENT MAP

General setup: $\left\{ \begin{array}{c} \text{circular} \\ \text{planar} \\ \text{networks} \end{array} \right\} \xrightarrow{\infty - 1} \left\{ \begin{array}{c} \text{points in} \\ \text{TNN} \\ \text{Grassmannian} \end{array} \right\}$



Today's special case: $\left\{ \begin{array}{c} \text{Le-tableaux/} \\ \Gamma\text{-networks} \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{points in} \\ \text{TNN} \\ \text{Grassmannian} \end{array} \right\}$

The boundary measurement matrix is similar in spirit to the weighted path matrices of the classical results of Lindström and Gessel-Viennot, but has some signs built in to the initial setup

- to account for directed cycles (in general, not today)
- to fit everything into the Grassmannian setting.

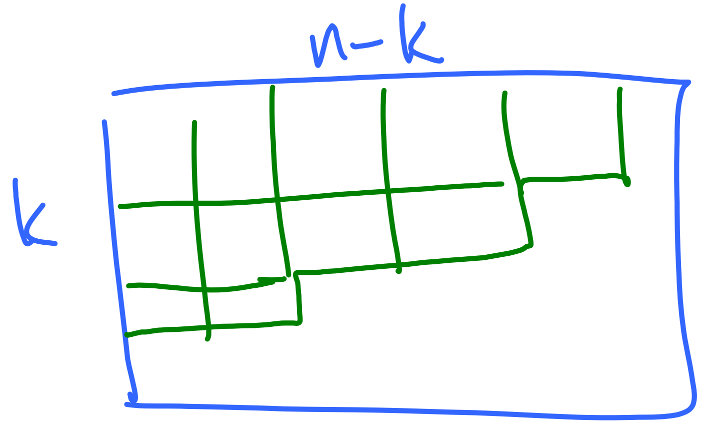
Upshot: "positive" formulas

3 steps:

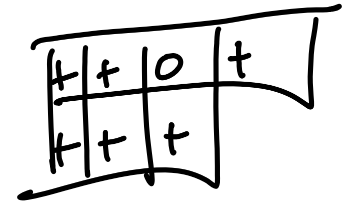
1) Le - diagrams
2) ⅂ - graphs and networks
3) Boundary meas. matrix

1) Le-diagrams $Gr(k,n)_{\geq 0}$

Young diagram in

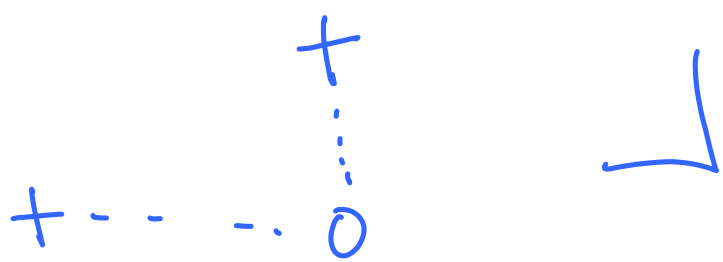


Eg

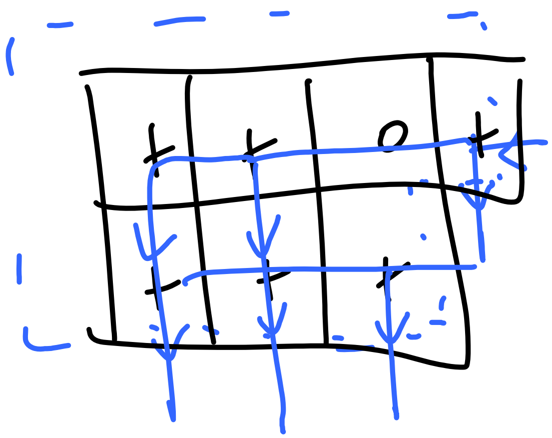


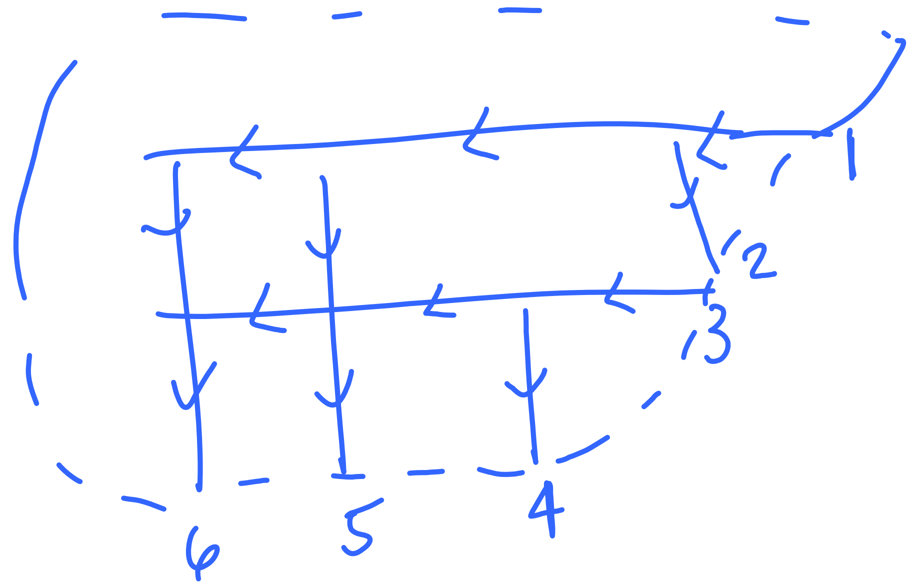
$\in Gr(2,6)_{\geq 0}$

Fill with + and 0 so that we avoid

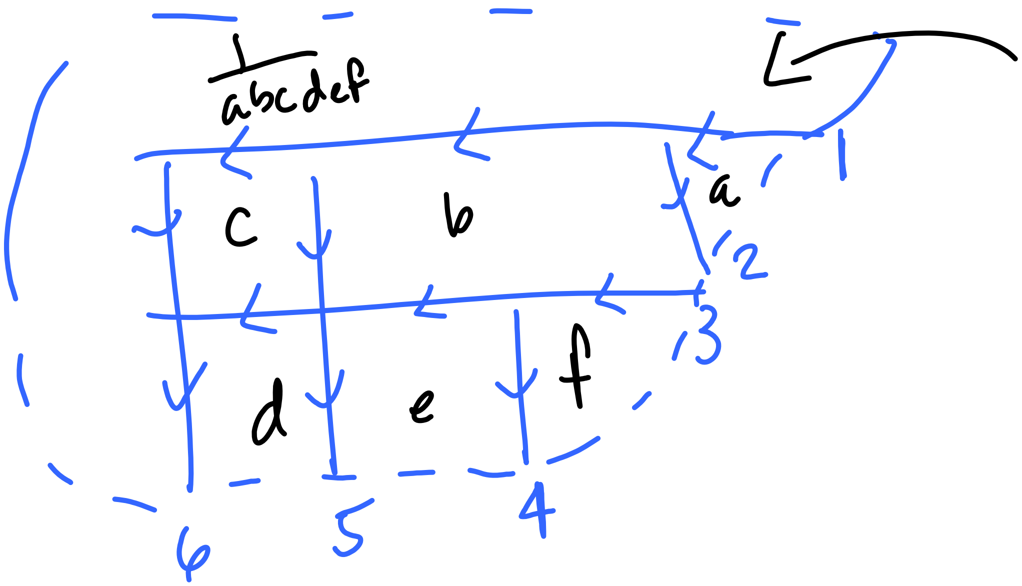


2) Γ-graphs from Le-diagrams





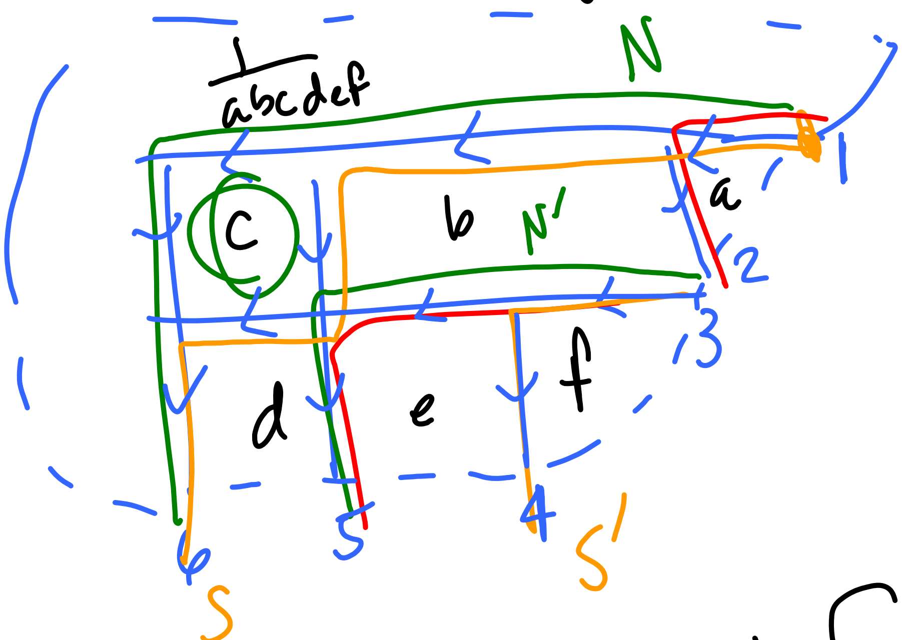
2.5) Γ-networks from Γ-graphs



1 condition on weights

$\prod$ all wts $= 1$

3) The boundary measurement matrix of a T-network.



$Gr(2,6)_{\geq 0}$

wt path = $\prod$ face weights below it

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
\begin{matrix} 1 \\ 3 \end{matrix} & \begin{bmatrix} 1 & a & 0 & 0 & -abef & -abdef(1+c) \\ 0 & 0 & 1 & f & ef & def \end{bmatrix}
\end{array}
$$

Thm: The boundary measurement matrix of a network with positive weights is TNN.

(General setting: recursive argument by Postnikov, formulas for minors by Talaska '08.)

For Γ-networks, each Δ_J is a weight generating function for families of non-intersecting paths connecting $I - J$ to $J - I$.

$\uparrow$ fixed source set

Thm: The map sending a Γ-network N to its boundary measurement matrix $A(N)$ is a bijection:

(Postnikov '07, Talaska '11)

$$\left\{ \begin{array}{c} \mathbb{R}_{>0}\text{-weightings of} \\ \text{a fixed } \Gamma\text{-graph } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{points in the} \\ \text{positroid cell corresp} \\ \text{to } G \end{array} \right\}$$

Ie:

1) different choices of weights give different points in $Gr(k,n)_{\geq 0}$

2) we get <u>every</u> point in the TNN Grassmannian this way.

So far, we looked at

$$\{ \mathrm{7}\text{-networks} \} \longrightarrow Gr(k,n)_{\geq 0}.$$

What about $\longleftarrow$?

3 stages: a) Determine shape of Le-diag

b) Determine underlying graph.

c) Write formulas for the weights.

("Big" cell: in terms of a nice cluster.)

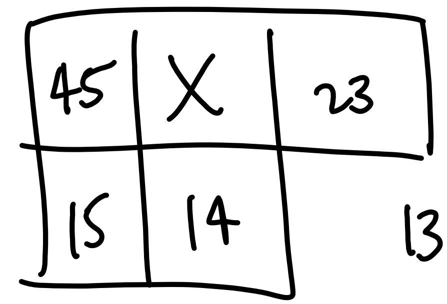
Eg: $Gr(2,5)_{\geq 0}$

$P_{12}=0$, $P_{13}=+$, $P_{14}=0$, $P_{15}=+$

$P_{23}=+$, $P_{24}=+$, $P_{25}=+$

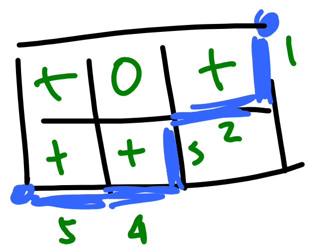
$P_{34}=0$, $P_{35}=+$

$P_{45}=+$



a) shape of partition: lex min'l is P_{13}

b)



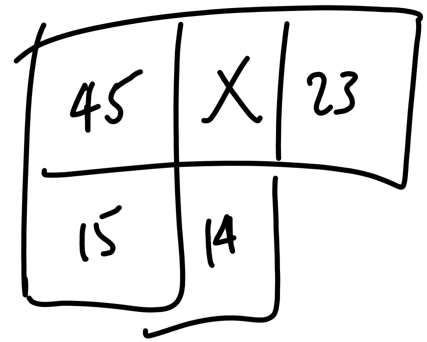
path $1 \to 2$?

$1,3 \to 2,3$

Δ_{23}

c) Figure out weights on the faces

3) weights on faces

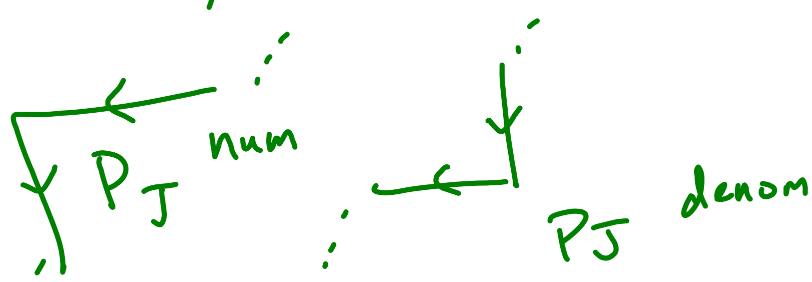


$$\frac{\text{corner}(N)}{\text{corner}(N')} \cdot \frac{\text{corner}(S')}{\text{corner}(S)}$$

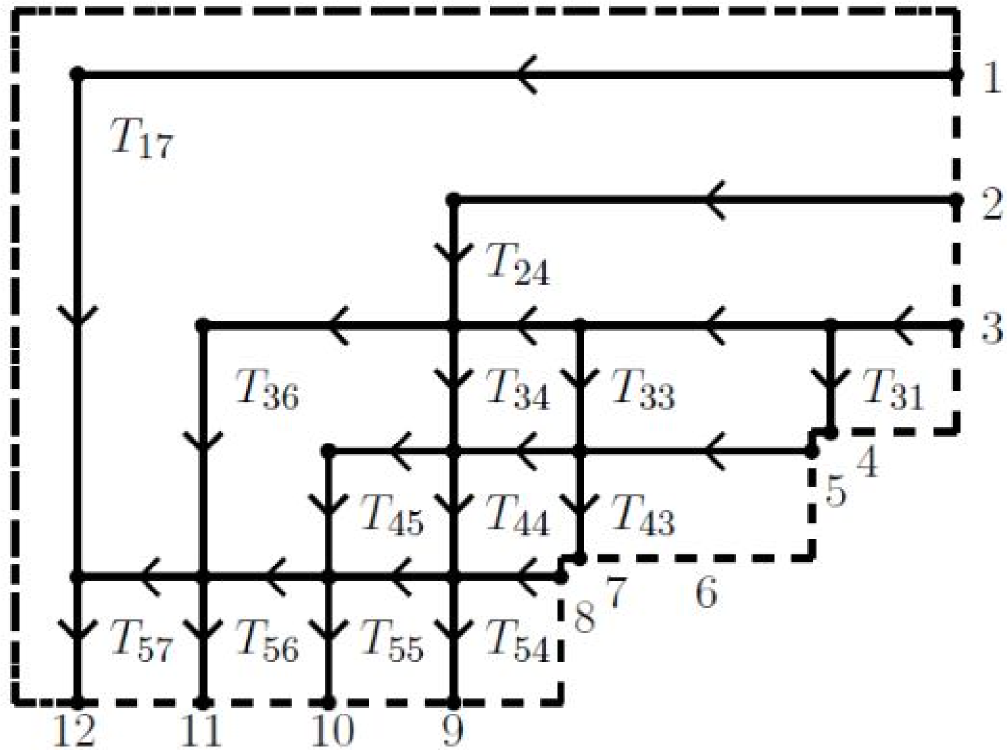
N north bdry of the face, N' next nested path under N

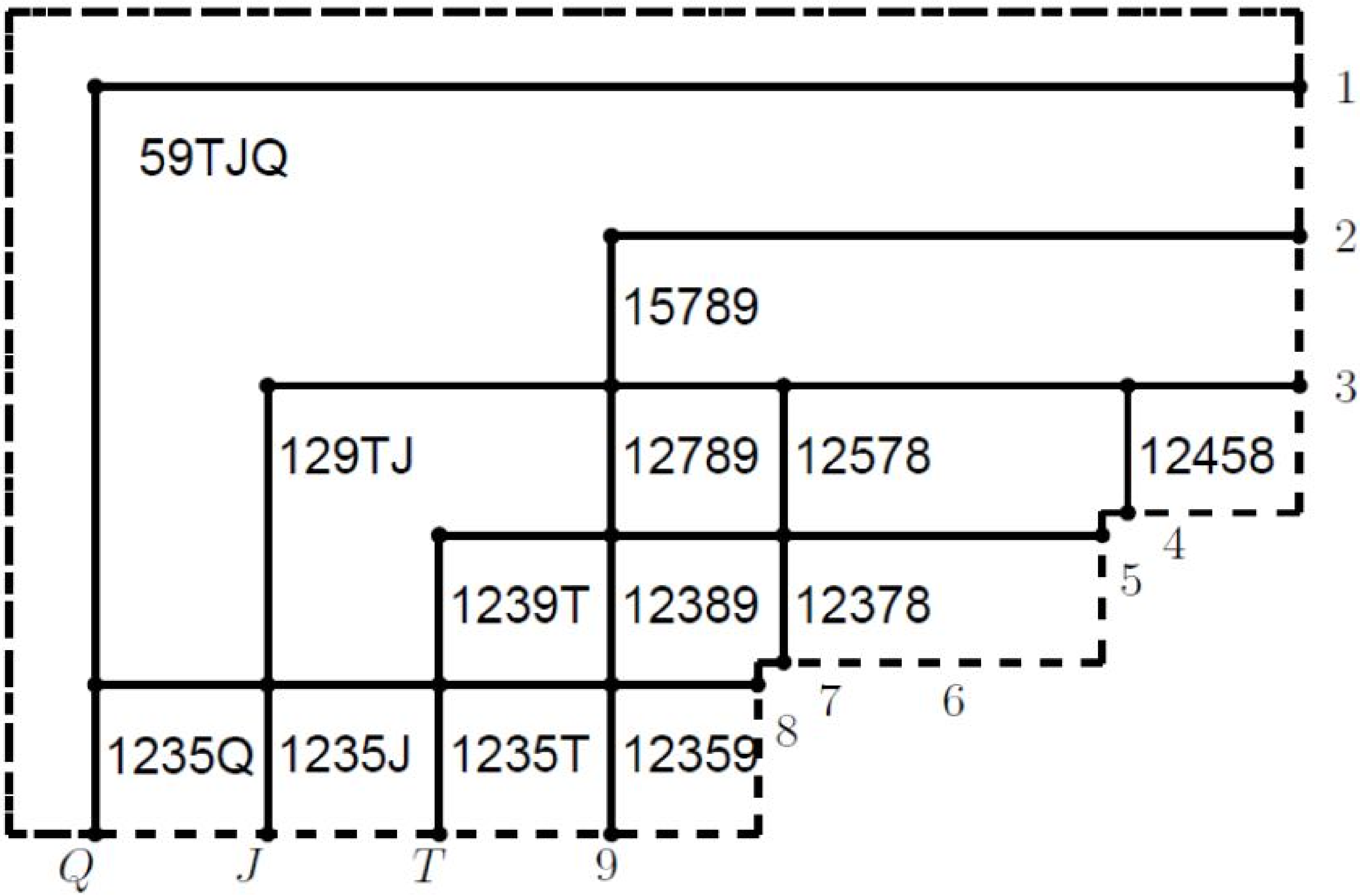
S south bdry of the face, S' nested under S.

corner records turns



T_{17}	0	0	0	0	0	0
0	0	0	T_{24}	0	0	0
0	T_{36}	0	T_{34}	T_{33}	0	T_{31}
0	0	T_{45}	T_{44}	T_{43}	0	
T_{57}	T_{56}	T_{55}	T_{54}			




1
59TJQ
2
15789
3
129TJ
12789
12578
12458
4
5
1239T
12389
12378
7
6
8
1235Q
1235J
1235T
12359
Q
J
T
9

Some connections:

- KP equation and solitons (next talk!)
- Recover total positive matrices (and their planar networks) as a special case sitting inside this Grassmannian set up.

$$\underset{\text{TNN}}{n \times n \text{ matrices}} \subseteq Gr(n, 2n)_{\geq 0}$$